REGULARITE EN CALCUL DES VARIATIONS. ESPACES DE SOBOLEV FRACTIONNAIRES.

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THESE

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Résumé

Dans cette thèse, on aborde plusieurs questions ayant trait au calcul des variations et à la théorie des équations aux dérivées partielles elliptiques.

Dans le Chapitre 1, on étudie la *condition de pente minorée* pour des fonctions définies sur la frontière d'un ouvert de $\mathbb{R}^n$. Cette condition, introduite dans [28], est une généralisation de la *condition de pente bornée* utilisée dans la théorie d'Hilbert-Haar, en calcul des variations . On met en évidence des propriétés de régularité des fonctions vérifiant la condition de pente minorée et on donne des conditions nécessaires et suffisantes sur les gradients d'une fonction pour qu'elle vérifie cette condition de pente minorée.

Dans le Chapitre 2, on s'intéresse à un problème de calcul des variations où la fonctionnelle est de la forme

$$u \mapsto \int \{F(\nabla u(x)) + G(x, u(x))\} \, dx$$

et où la condition de Dirichlet est définie par une fonction vérifiant la condition de pente minorée. On montre que toute solution est localement lipschitzienne. Ceci généralise le résultat de [28] obtenu quand $G = 0$.

Dans le Chapitre 3, on étudie une équation aux dérivées partielles elliptique à forme divergentielle avec une condition de Dirichlet qui vérifie la condition de pente minorée. On prouve l'existence d'une solution localement lipschitzienne. Ceci généralise certains résultats de [47] où la condition de Dirichlet vérifiait la condition de pente bornée.

Dans le Chapitre 4, on décrit les composantes connexes de l'ensemble $W^{s,p}(M, N)$, où M et N sont deux variétés compactes, $0 < s < 1 + 1/p$. Ceci généralise les résultats obtenus pour $s = 1$ dans [89], [20], [40].

Dans le Chapitre 5, nous identifions l'ensemble singulier d'une fonction $u \in W^{s,p}(S^N, S^1)$ quand $s \geq 1$. Ce résultat était connu uniquement quand $s \leq 1$ et $sp = 1$ (voir [10], [22], [2]).

Contents

Introduction

0.1 Existence et régularité en calcul des variations

Dans la première partie de cette thèse, on s'intéresse à la question de la régularité dans un problème de calcul des variations général, lorsque les fonctions admissibles sont définies sur un ouvert de $\mathbb{R}^n, n \geq 2$ et à valeurs dans $\mathbb{R}$. Plus précisément, étant donné une fonction

$$L : (x, z, \kappa) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

(le lagrangien) et un ouvert Ω borné de $\mathbb{R}^n$, on s'intéresse au problème de minimiser la fonctionnelle suivante :

$$I : u \to \int_\Omega L(x, u(x), \nabla u(x))\, dx$$

sur un ensemble de fonctions $u \in \mathcal{E}$ (l'ensemble des fonctions admissibles). Une solution de ce problème est donc une fonction $\bar{u} \in \mathcal{E}$ telle que $I(\bar{u}) \leq I(u)$ pour tout $u \in \mathcal{E}$.

Le choix de l'ensemble $\mathcal{E}$ dépend de deux types de contraintes : d'une part, l'origine du problème (modélisation d'une situation physique, question intermédiaire d'un autre problème mathématique...) et d'autre part, les possibilités mathématiques du problème lui-même (ce qu'on peut faire pour cette fonctionnelle avec les outils mathématiques dont on dispose). Ces contraintes conditionnent notamment le choix des conditions au bord satisfaites par les fonctions admissibles et le type de régularité des fonctions admissibles.

Idéalement, on aimerait prouver l'existence (voire l'unicité) d'une solution C^1 sur l'ouvert Ω, de manière à donner un sens "maximal" à l'écriture ∇u. Par ailleurs, dans toute cette thèse, les conditions au bord seront de type Dirichlet, c'est-à-dire qu'on se donnera une fonction ϕ définie sur le bord Γ de Ω à valeurs dans $\mathbb{R}$ et les fonctions admissibles devront être égales à ϕ sur Γ. Pour donner un sens fort à cette "égalité au bord", on pourrait espérer que les fonctions admissibles soient continues sur l'adhérence de l'ouvert.

Malheureusement, le problème considéré n'a pas en général de solution C^1 sur l'ouvert et continue sur l'adhérence de l'ouvert. On doit donc donner un sens affaibli au mot solution et élargir l'ensemble $\mathcal{E}$. La "voie royale" pour prouver l'existence d'une solution est la *méthode directe* (voir [29]). On peut considérer qu'Hilbert est le premier à l'utiliser pour donner une preuve rigoureuse du principe de Dirichlet (voir l'introduction de [37] et les références qui y sont données). Pour être appliquée, cette méthode recquiert d'une part que la fonctionnelle I soit semicontinue inférieurement et minorée, d'autre part que l'ensemble $\mathcal{E}$ ait de bonnes propriétés de compacité. Ces deux propriétés ont tendance à s'opposer : "plus la topologie est faible, plus les suites convergeront facilement "; autrement dit, une topologie faible favorise les propriétés de compacité et réduit le nombre de fonctionnelles semicontinues inférieurement.

Dans le cas d'un problème où $n = 1$, Tonnelli a mis en évidence le "bon" espace $\mathcal{E}$ pour établir un théorème d'existence général : il s'agit de l'ensemble des fonctions absolument continues. Dans le cas $n \geq 2$, le "bon" espace des fonctions admissibles apparaît indépendamment dans des articles de Sobolev et Calkin, Morrey : l'espace des fonctions de Sobolev, généralisation en un certain sens des fonctions absolument continues (voir néanmoins les références antérieures citées dans [71], section 1.8 "Lower semicontinuity").

Pour $p \geq 1$, on note $W^{1,p}(\Omega)$ l'espace de Sobolev constitué des fonctions $u : \Omega \to \mathbb{R}$ telles que $u \in L^p(\Omega)$ et dont les dérivées au sens des distributions sont également des fonctions de L^p. Cet ensemble est un espace de Banach pour la norme :

$$||u||_{W^{1,p}(\Omega)} := ||u||_{L^p(\Omega)} + \sum_{i=1}^{n} ||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)}.$$

On considèrera également l'espace $W_0^{1,p}(\Omega)$, défini comme l'adhérence dans $W^{1,p}(\Omega)$ des fonctions C^∞ à support compact dans Ω. Deux possibilités pour définir $\mathcal{E}$ s'ouvrent alors. Soit la fonction ϕ (définissant la condition de Dirichlet) peut s'étendre en un élément de $W^{1,p}(\Omega)$ encore noté ϕ et on peut alors choisir :

$$\mathcal{E} := W_0^{1,p}(\Omega) + \phi.$$

Soit l'ouvert Ω est régulier (au moins lipschitz) de sorte qu'on peut définir la notion de *trace* tr, prolongement à $W^{1,p}(\Omega)$ de la fonction "restriction à Γ" définie sur $C^0(\bar{\Omega})$. On définira alors

$$\mathcal{E} := \{u \in W^{1,p}(\Omega) : \mathrm{tr}\, u = \phi\}.$$

Cet ensemble est non vide si et seulement si $\phi \in L^1(\Omega)$ pour $p = 1$ et si $\phi \in W^{1-1/p,p}(\Gamma)$ pour $p > 1$ (voir [32]).

De tels espaces de fonctions admissibles présentent plusieurs intérêts. En premier lieu, ils permettent sous des hypothèses minimales sur L de rendre la fonction

$$x \to L(x, u(x), \nabla u(x))$$

intégrable et donc de donner un sens au problème de minimisation. Par ailleurs, sur le plan de l'analyse fonctionnelle, ces espaces sont réflexifs (pour $p > 1$), ont des propriétés de compacité (théorème de Rellich) et peuvent s'identifier à des sous-espaces de l'ensemble des fonctions continues (pour $p > n$) (théorème de Morrey).

Les propriétés de compacité et de réflexivité permettent en particulier de mettre en oeuvre la méthode directe d'existence en calcul des variations : on se donne une suite minimisante de fonctions admissibles. Si le lagrangien L vérifie une hypothèse de coercivité, cette suite est bornée dans $W^{1,p}$. On peut en extraire une sous-suite convergeant faiblement dans $W^{1,p}$ (réflexivité) et fortement dans L^p (théorème de Rellich). Enfin, un argument de semicontinuité inférieure (si L vérifie une hypothèse de convexité) permet de conclure que la limite de cette sous-suite est une solution du problème.

Une solution faible ainsi obtenue, il est naturel de se demander si elle ne serait pas un peu plus forte que prévue, c'est-à-dire continue ou même différentiable (au sens classique) sur tout ou partie de Ω. On a ainsi divisé en deux étapes (existence puis régularité *a posteriori*) la question initiale.

La théorie de la régularité prend son essor au début du siècle dernier (pour un bref historique, voir [71]). En 1912, Lichtenstein montre qu'une solution de classe C^2 d'un problème d'intégrale double régulier où L est analytique, est de classe C^3, et donc analytique par un théorème de Bernstein. En 1929, Hopf montre que la même conclusion a lieu lorsqu'on suppose seulement la solution de classe $C^{1,\mu}$, puis Morrey améliore encore ce résultat en montrant qu'il suffit de prouver que la solution est lipschitzienne. Hormis le cas $n = 2$ traité par Morrey et certain lagrangiens quadratiques considérés par Hirschfeld, il restait néanmoins un "vide" entre les solutions de type Sobolev données par les méthodes directes et la régularité lipschitzienne ou hölderienne requise par les théorèmes précédents. Deux voies s'ouvrent alors, que l'on peut désigner pour simplifier sous les noms de "théorie de De Giorgi" et "théorie de Hilbert-Haar".

0.2 La théorie de De Giorgi

Même si les résultats obtenus dans cette thèse ne s'inscrivent pas dans cette théorie, celle-ci a fait émerger des idées et des techniques abondamment utilisées dans les Chapitres 2 et 3. De plus, les premiers articles apparaissant après-guerre et appartenant à la théorie Hilbert-Haar sont redevables à plusieurs résultats dus à De Giorgi et aux nombreux auteurs qui ont amélioré et généralisé ses résultats (notamment Ladyzhenskaya et Ural'tseva, Moser, Morrey, etc...).

Dans la deuxième moitié des années cinquante, De Giorgi [30] et Nash [74] font paraître presque simultanément deux théorèmes semblables. Les travaux du premier auteur auront des répercussions importantes dans de nombreux articles sur la régularité en calcul des variations. On en verra plusieurs exemples dans la suite.

A l'exception de la méthode directe de régularité en calcul des variations dont les principaux inspirateurs sont Giaquinta et Giusti, la grande majorité des résultats obtenus dans le cadre de la théorie de De Giorgi sont des conséquences de théorèmes de régularité pour les équations elliptiques linéaires ou quasilinéaires. C'est déjà le cas dans l'article originel de De Giorgi : le problème est de minimiser :

$$I : u \mapsto \int_\Omega F(\nabla u)$$

sur $W^{1,2}(\Omega)$. (Lorsque le lagrangien ne dépend que de la variable κ, on notera $F(\kappa) = L(x, z, \kappa)$). On suppose que F est $C^2(\mathbb{R}^n)$ et qu'il existe $0 < \mu < C$ tel que

$$\mu|\xi|^2 \le \langle \nabla^2 F(\kappa)\xi, \xi\rangle \le C|\xi|^2 \ , \ \kappa \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n. \tag{1}$$

Dans la suite de cette introduction, on dira que (1) constitue une hypothèse de croissance sur F, à la fois minorante (c'est l'inégalité de gauche) et majorante (c'est l'inégalité de droite).

Le théorème de De Giorgi affirme que si u est un minimiseur local, i.e.

$$I(u) \leq I(u+\theta) \quad \forall \theta \in C_c^\infty(\Omega),$$

alors la restriction de u à tout compact contenu dans Ω est $C^{1,\alpha}$ pour un certain $\alpha \in (0,1)$ qui ne dépend pas du compact.

Sous les hypothèses (1), un minimiseur local u est solution de l'*équation d'Euler sous forme faible* :

$$\int_\Omega \langle \nabla F(\nabla u(x)), \nabla \theta(x) \rangle \, dx = 0 \quad \forall \theta \in C_c^\infty(\Omega). \tag{2}$$

Réciproquement, supposons que F est convexe, que u est solution de (2) et appartient à $W^{1,2}(\Omega)$. Alors u est un minimiseur local (cela résulte du fait qu'un point annulant la dérivée d' une fonction convexe est un minimiseur global de cette fonction).

La méthode de De Giorgi est une méthode indirecte, puisqu'elle passe par l'équation d'Euler-Lagrange pour obtenir de la régularité. C'est une méthode locale au sens où la régularité sera toujours démontrée au voisinage d'un point et en recouvrant $\bar{\Omega}$ par de tels voisinages. Le théorème originel de De Giorgi a été amélioré dans plusieurs directions. D'abord, une preuve plus simple a été proposée par Moser. Néanmoins, les ensembles fonctionnels introduits par De Giorgi ont eu une belle postérité (on les retrouve par exemple dans [37]). Ensuite, le théorème a été étendu à des lagrangiens dépendant de x et z, avec des hypothèses de croissance très variées. On pourra se reporter aux livres [71], [36] et [53].

On ne sait pas démontrer en général qu'une solution d'un problème variationnel est solution de l'équation d'Euler correspondante. La difficulté consiste à vérifier les hypothèses du théorème de dérivation sous le signe intégral. Ces hypothèses sont satisfaites dans deux cas principaux. Le premier est celui où on dispose d'hypothèses de croissance (majorantes et minorantes) du lagrangien, comme dans (1). Le second cas est celui où on sait *a priori* que u est lipschitzien, ce qui implique que les hypothèses de croissance précédentes sur le lagrangien évalué en u sont automatiquement satisfaites. (Sur ce sujet, on pourra consulter le cas particulier envisagé dans [25]) .

Pour obtenir de la régularité sur un problème non linéaire (problème de minimisation d'une fonctionnelle), De Giorgi le transforme en un problème linéaire (équation elliptique linéaire à forme divergentielle et sans second membre). Le coût de ce procédé est de devoir supposer des hypothèses de croissance (minorantes et majorantes) sur le lagrangien pour pouvoir obtenir une borne L^∞ sur les coefficients de l'équation linéaire. Cette transformation résulte de l'application de la méthode des quotients différentiels à l'équation d'Euler sous forme faible. L'objectif principal de la méthode des quotients différentiels est d'obtenir des informations sur les dérivés secondes d'une solution (par exemple montrer que cette solution est dans $W^{2,2}_{\mathrm{loc}}$). Comme on ne sait pas *a priori* si ces dérivés existent, on remplace la dérivée dans une direction par une dérivée discrète. Ainsi, si u est solution et qu'on s'intéresse à $\dfrac{\partial \nabla u}{\partial x_1}$, on introduira

$$\Delta_{t,1} \nabla u(x) := \frac{\nabla u(x+te_1) - \nabla u(x)}{t}.$$

La méthode des quotients différentiels est aussi bien utilisée dans la théorie des équations elliptiques (voir [36], [16]) que dans celle de la régularité en calcul des variations (voir [71], [37] et aussi [58] où elle est utilisée de manière particulièrement astucieuse).

D'un point de vue technique, dans la théorie de De Giorgi, on s'efforce en général d'estimer la mesure de Lebesgue d'ensembles de niveau d'une solution u : $|\{x \in \Omega : u(x) \geq k\}|$ ou de $f(u)$ pour une fonction f bien choisie (souvent une fonction puissance, ou une fonction "troncature" $\max(u,k)$). Cette estimation a souvent lieu en termes de normes de ∇u, notamment $||\nabla u||_{L^p}$. L'inégalité de Sobolev y joue un rôle fondamental.

Rappelons ici cette inégalité. Pour tout $u \in W_0^{1,p}(\Omega)$, $p < n$, on a

$$||u||_{L^{p^\star}(\Omega)} \leq C||\nabla u||_{L^p(\Omega)} \ , \quad (\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{n}).$$

L'inégalité de Sobolev permet en particulier un gain d'intégrabilité sur la fonction lorsqu'on peut contrôler son gradient. Elle est à l'origine de la méthode d'itération de Moser (voir [72], et aussi [36], [59]).

La manière principale de trouver des propriétés d'une solution d'un problème de calcul des variations est de comparer cette solution avec d'autres fonctions admissibles construites à partir de la solution elle-même. De même, pour établir des propriétés de solution d'équations elliptiques écrites sous forme faible (i.e. où les dérivées d'ordre le plus élevé sont reportées sur les fonctions test), la voie principale consiste à choisir de bonnes fonctions test, construites à partir de ces solutions. Par exemple, la fonction $\max(u-k,0)\eta^2$ où η est une fonction *cutt off* et $k \in \mathbb{Z}$, n'est pas loin d'apparaître dans chacun des articles cités dans cette section. Elle sert notamment à majorer u.

Une idée proche consiste à se demander quelles sont les propriétés vérifiées par $f(u)$ lorsque u est solution d'une équation ou d'un problème variationnel. Par exemple, l'outil principal dans la simplification par Moser [72] de la preuve du théorème originel de De Giorgi est de considérer des fonctions de la forme $f(u)$ lorsque u est solution, et f une fonction positive convexe.

Au début des années 80, des avancées notables ont été accomplies dans l'affaiblissement des hypothèses de différentiabilité sur le Lagrangien. Ainsi, la démonstration de De Giorgi a inspiré la méthode directe en régularité de Giaquinta et Giusti. Par exemple, dans l'article [33], les auteurs montrent la continuité hölderienne locale des solutions de problèmes variationnels. Le fait remarquable et nouveau dans cet article est qu'on ne suppose aucune hypothèse de différentiabilité sur L, car on n'utilise pas l'équation d'Euler (c'est ce qui justifie l'appellation *méthode directe en régularité*). On ne suppose pas non plus L convexe. En revanche, dans cet article (comme dans tous les travaux évoqués dans ce paragraphe), des hypothèses de croissance sont imposées à L, en l'occurence :

$$|\kappa|^m - k \leq L(x,z,\kappa) \leq a|\kappa|^m + k.$$

Giaquinta [37] a proposé un contrexemple (parallèlement à Marcellini) pour montrer que la fonction majorante de L doit être du même ordre que la fonction minorante si l'on veut obtenir de la régularité. Ce lagrangien est $F(\kappa) =$

$\sum_{i=1}^{n-1} \kappa_i^2 + \kappa_n^4/2$. Un minimiseur pour $n \geq 6$ et Ω la boule unité est donné par

$$u(x) := \frac{\sqrt{n-4}}{24} \frac{x_n^2}{\sqrt{\sum_{i=1}^{n-1} x_i^2}}.$$

Notons que la trace de ce minimiseur sur la sphère n'est même pas continue. C'est également le cas des variantes de ce contrexemple ([58], [50]), qui proposent un lagrangien uniformément elliptique.

Il n'en reste pas moins que beaucoup d'articles ont proposé un ensemble d'hypothèses du type "p, q growth condition", c'est-à-dire

$$m|\xi|^p \leq F(\xi) \leq M(1 + |\xi|^q)$$

ou des conditions anisotropes (pour une bibliographie récente sur le sujet, voir [60]) qui permettent d'obtenir qu'un minimiseur d'un problème variationnel ou une solution d'équation elliptique sont localement lispchitziens.

0.3 La théorie de Hilbert-Haar

0.3.1 Enoncé classique

Dans l'énoncé classique du théorème de Hilbert-Haar, tel qu'il est formulé au milieu des années 60, on considère un lagrangien de la forme $L(x, u, \kappa) = F(\kappa)$, convexe sur $\mathbb{R}^n$. On suppose que la condition au bord $\phi : \Gamma \to \mathbb{R}$ vérifie une condition de pente bornée (qu'on définira ultérieurement). Alors le théorème affirme qu'il existe une fonction lipschitzienne u qui minimise I sur l'ensemble des fonctions lipschitziennes qui valent ϕ au bord.

Ainsi énoncé, le théorème de Hilbert-Haar apparaît à la fois comme un théorème d'existence et de régularité. En effet, les résultats d'existence obtenus par la méthode directe assertent l'existence d'une solution dans un ensemble de fonctions de type Sobolev. Ici, on obtient une solution dans un espace de fonctions beaucoup plus régulières. Non seulement les fonctions lipschitziennes sont dérivables presque partout, ce qui donne un sens fort à ∇u dans l'écriture de la fonctionnelle I, mais surtout, un minimiseur lipschitzien vérifie l'équation d'Euler-Lagrange (2), pour peu que le lagrangien F soit différentiable. C'est une première différence essentielle par rapport à la méthode de De Giorgi, qui exigeait des hypotèses de croissance sur le lagrangien pour obtenir l'équation d'Euler-Lagrange. En revanche, une fois l'équation d'Euler-Lagrange établie, la méthode de De Giorgi prend le relais de la théorie Hilbert-Haar pour établir la régularité hölderienne, voire analytique, d'un minimiseur. La complémentarité des deux méthodes est notamment explicite dans le livre [65].

Contrairement à la méthode de De Giorgi, le minimiseur obtenu ici est global (c'est-à-dire que $I(u) \leq I(u + \theta)$ pour tout θ lipschitzien valant 0 au bord et pas seulement pour tout $\theta \in C_c^\infty(\Omega)$). Cela est dû à l'hypothèse essentielle qu'on s'est donnée : ϕ vérifie la condition de pente bornée. D'une certaine manière, la régularité qu'on impose au minimiseur à la frontière se propage à l'intérieur de l'ouvert. Cette propriété contraste fortement avec la notion de régularité locale qui apparaît dans la méthode de De Giorgi et la théorie des équations elliptiques (voir par exemple [16], Remarque IX.26).

0.3.2 Bref historique

Le théorème de Hilbert-Haar, tel qu'on l'a énoncé dans la section précédente, est le produit de très nombreuses contributions. Le seul énoncé dont on disposait avant guerre, comme on peut le lire dans le livre de Morrey [71], concerne le cas $n = 2$ et un lagrangien qui ne dépend que de κ. Il affirme qu'il existe une unique fonction minimisante définie sur un domaine strictement convexe sous la condition des trois points. Cette condition est l'existence d'une constante K bornant la pente de tout plan défini par la donnée de trois points de la forme $(x, \phi(x))$ avec $x \in \Gamma$. Il est équivalent à la condition de pente bornée (définie dans la section suivante) pour $n = 2$.

Il semble que ce soit l'article de De Giorgi [30] qui ait inspiré les premières versions du théorème de Hilbert-Haar en dimension finie quelconque. A ma connaissance, la première de ces versions apparaît dans l'article de Stampacchia [84] qui utilise explicitement le théorème de De Giorgi et certaines de ses généralisations. Mais Stampacchia impose à F d'être de classe C^2 et elliptique (au sens où les valeurs propres de sa hessienne sont > 0 en tout point) et il suppose de plus que Ω est de classe $C^{1,1}$ et que ϕ satisfait la condition de pente bornée et est la trace d'une fonction de $W^{2,p}(\Omega), p > n$.

Dans son livre [71], Morrey présente une amélioration du résultat de Stampacchia et livre un énoncé semblable à celui qu'on a donné au début de la section 0.3.1, à ceci près que Ω est supposé strictement convexe (cela dit, on n'a pas su voir dans la preuve où cette hypothèse était déterminante). La référence à la théorie de De Giorgi reste évidente. Il semble que Morrey n'ait pas eu connaissance de l'article de Miranda [69] qui, le premier, donne son autonomie à la théorie de Hilbert-Haar par rapport à la théorie de De Giorgi. Pour la première fois également, on s'aperçoit que la convexité de l'ouvert n'a pas besoin d'être une hypothèse explicite. Cela dit, Miranda suppose encore son lagrangien C^2 et strictement convexe, même si ces hyptohèses ne sont pas incontournables dans sa preuve. (En fait, Miranda s'intéresse particulièrement au cas du lagrangien $F(\kappa) := \sqrt{1 + |\kappa|^2}$ qui ne peut être traité par le théorème de De Giorgi, et qui est évidemment C^2 et strictement convexe). La démonstration de Miranda est fondée sur une idée utilisée avant guerre, qui est une forme de principe du maximum sur le gradient. On y reviendra en détails dans la section 0.3.6.3.

Hartman et Stampacchia [47] s'inspirent de cette même idée pour obtenir des théorèmes d'existence dans l'ensemble des fonctions lipschitziennes pour des équations elliptiques. Il propose aussi la version finale du théorème de Hilbert-Haar (i.e. celle du début de la section 0.3.1).

L'histoire du théorème de Hilbert-Haar paraît s'assoupir pendant plus de vingt-cinq ans. Cependant, au début des années 2000, un article de Cellina [24] introduit le théorème de Hilbert-Haar dans un contexte différent et en modifie la philosophie. De théorème d'existence dans l'ensemble des fonctions lipschitziennes, le problème devient une question de régularité pure : quand un minimiseur dans l'ensemble des fonctions de type Sobolev est-il lipschitzien ? De plus, le cas de lagrangiens à valeurs éventuellement $+\infty$ est envisagé pour la première fois. L'irruption des espaces de Sobolev dans la théorie est consacrée ensuite par trois articles de Mariconda et Treu [62], [61], [63] (voir aussi [85]). L'article [61] répond à la question de Cellina tandis que [62] réécrit le premier Chapitre de [37] dans le cadre des espaces de Sobolev, en utilisant un langage très éclairant pour signifier les inégalités vérifiées presque partout à la frontière par des fonctions de type Sobolev. Ce langage apparaît déjà chez [84] et de manière plus importante encore dans le livre [36].

Enfin, la contribution la plus récente dans la théorie Hilbert-Haar apparaît dans l'article de Clarke [28]. Tout en utilisant les développements récents, elle insuffle plusieurs idées nouvelles qui ont largement inspiré les Chapitres 2 et 3 de cette thèse. Pour comprendre l'intérêt de cette contribution, et celui de ces deux chapitres, il importe de faire deux reproches au théorème de Hilbert-Haar traditionnel. D'abord, la condition de pente bornée peut être très restrictive, comme on le verra dans la section suivante. Ensuite, la forme du lagrangien (qui ne dépend que de la variable κ) est également très restrictive. On y reviendra dans la section 0.3.4.

0.3.3 La condition de pente bornée

On dit que ϕ vérifie la condition de pente bornée de constante $Q > 0$ si pour tout $y \in \Gamma$, il existe $\zeta_y^{\pm} \in R^n, |\zeta_y^{\pm}| \leq Q$ tels que :

$$\phi(y) + \langle \zeta_y^-, x - y \rangle \leq \phi(x) \leq \phi(y) + \langle \zeta_y^+, x - y \rangle \quad \forall x \in \Gamma.$$

Ainsi, la condition de pente bornée signifie qu'on peut encadrer la fonction ϕ sur Γ par deux familles de fonctions affines, dont chaque élément coïncide avec ϕ en un point de Γ. Historiquement, il semble que cette condition apparaisse avant guerre dans le cas $n = 2$ dans les travaux de Rado, puis seulement après guerre, en dimension finie quelconque dans un article d'Hartman et Niremberg [46]. La terminologie suivante y est employée. On considère $\mathbb{R}^n \times \mathbb{R}$ muni des coordonnées $(x_1, .., x_n; z)$ et on appelle l'axe des z l'axe vertical. On dit qu'un sous-ensemble A_1 de $\mathbb{R}^n \times \mathbb{R}$ est au-dessus d'un sous-ensemble A_2 si pout tout couple $(x; z_1) \in A_1$, $(x; z_2) \in A_2$, ayant la même "abscisse" $x \in \mathbb{R}^n$, on a l'inégalité $z_1 \geq z_2$. On définit symétriquement la notion "être au-dessous".

La pente d'une hypersurface de $\mathbb{R}^n \times \mathbb{R}$ est la valeur absolue de la tangente de l'angle entre sa normale et l'axe vertical. Pour donner une expression analytique de cette pente, on peut considérer une hypersurface d'équation $z = z(x), x \in D$ où D est un ouvert de $\mathbb{R}^n$ et z est une fonction C^1. Une normale unitaire à cette hypersurface en $(x, z(x))$ est

$$\left(\frac{\nabla z}{\sqrt{1 + |\nabla z|^2}}, \frac{-1}{\sqrt{1 + |\nabla z|^2}}\right).$$

Si θ désigne l'angle entre la normale et l'axe vertical, on a

$$|\cos\theta| = 1/\sqrt{1 + |\nabla z|^2}$$

et la pente est donc par définition $|\tan\theta| = \sqrt{1 - \cos^2\theta}/|\cos\theta| = |\nabla z|$.

Dans l'article d'Hartman et Niremberg [46], on considère un ouvert convexe borné Ω de $\mathbb{R}^n$ et une fonction continue z sur $\bar{\Omega}$. Cette fonction z définit l'hypersurface S d'équation $z = z(x), x \in \bar{\Omega}$ et S' sa frontière. On introduit l'hypothèse suivante : il existe $Q > 0$ tel que par tout point de S', il passe deux hyperplans de pente Q tel que S' est au-dessous de l'un et au-dessus de l'autre. Explicitons cette propriété : soit $y \in \Gamma = \partial\Omega$. Alors $(y, z(y)) \in S'$. Il existe deux hyperplans $H_y^{\pm}$ d'équation $z = z^{\pm}(x)$, avec $|\nabla z^{\pm}| \leq Q$ qui coïncident avec S' en $(y, z(y))$ et qui encadrent S' au sens

$$z^-(x) \leq z(x) \leq z^+(x) \quad \forall x \in \Gamma.$$

Ceci est exactement la condition de pente bornée (même si cette appellation n'apparaît pas dans l'article). Ainsi, en des termes géométriques, la condition de pente bornée signifie que la 'courbe'

$$\{(x, \phi(x)) \in \mathbb{R}^n \times \mathbb{R} : x \in \Gamma\}$$

est située entre deux familles d'hyperplans $(H_y^{\pm})$ de $\mathbb{R}^n \times \mathbb{R}$ dont on peut borner la pente uniformément.

Le théorème d'Hartman et Niremberg ne s'inscrit pas dans un contexte de calcul des variations (il affirme que si z est C^2 sur Ω et C^1 sur $\bar{\Omega}$, si la hessienne de z possède des valeurs propres positives et négatives en tout point de Ω et a un déterminant qui ne change pas de signe sur Ω, alors sous l'hypothèse précédente sur ϕ, on a $|\nabla z| \leq Q$ sur Ω).

L'expression *bounded slope condition* apparaît pour la première fois dans [84]. Le premier résultat sur la bounded slope condition est affirmé par Gilbarg [35] et démontré par Miranda [69]. Il s'énonce ainsi : la restriction d'une fonction C^2 à la frontière d'un ouvert uniformément convexe vérifie une condition de pente bornée. Un ouvert (qui n'est pas nécessairement de classe C^2) est dit uniformément convexe s'il existe $\epsilon > 0$ tel que pour tout $x \in \Gamma$, il existe un vecteur unitaire n_x tel que pour tout $y \in \Omega$,

$$\langle n_x, y - x \rangle \geq \epsilon |y - x|^2. \tag{3}$$

Il semble qu'il faille attendre les travaux d'Hartman pour qu'on prenne conscience que la convexité de l'ouvert Ω est une hypothèse déjà contenue dans la condition de pente bornée. Plus précisément, si la fonction ϕ n'est pas affine et vérifie la condition de pente bornée, alors l'ouvert Ω est convexe. Les deux articles d'Hartman [43] et [45] contiennent toutes les propriétés connues des fonctions vérifiant une condition de pente bornée.

Dans l'article *On the bounded slope condition* de 1966 [43], trois résultats importants sont avancés.

Le premier relie une propriété vérifiée par ϕ sur le bord Γ de Ω à une propriété de convexité pour des fonctions $\phi^{\pm}$ définie sur $\mathbb{R}^n$. Etant donné $\bar{x} \in \Omega$ et $t > 0$, on définit pour $x \in \mathbb{R}^n$

$$\phi^{\pm}(x) = \pm\theta t + (1 - \theta)\phi(y) \tag{4}$$

où $\theta \in [0, +\infty), y \in \Gamma$ sont définis par $x = \theta\bar{x} + (1 - \theta)y$. En des termes plus géométriques, le graphe de la fonction ϕ^+ est un cône de sommet $(\bar{x}, t)$ et qui "s'appuie sur le graphe de ϕ". Celui de ϕ^- est son "symétrique". Le premier résultat d'Hartman affirme que ϕ vérifie une condition de pente bornée si et seulement si pour tout $\bar{x} \in \Omega$, il existe $N > 0$ tel que pour tout $t > N$, les fonctions ϕ^- et $-\phi^+$ sont convexes.

Ainsi, non seulement une fonction satisfait la condition de pente bornée si et seulement si c'est la restriction d'une fonction convexe et d'une fonction concave (ce qu'avait déjà constaté Morrey, voir le Lemme 4.2.3 de [71]), mais de plus, on peut choisir ces deux fonctions convexes et concaves d'une forme particulière : elles sont affines sur les droites passant par $\bar{x}$, un point arbitraire de Ω. Ce fait est essentiel dans [43].

En utilisant le fait qu'une fonction est convexe si et seulement si sa restriction à toute droite est convexe, Hartman donne un critère pratique pour savoir si ϕ

vérifie ou non la condition de pente bornée. Plus précisément, si on fixe $\bar{x} \in \Omega$, il s'agit de trouver $N > 0$ tel que pour tout $|t| \geq N$, pour tout triplet de points de Γ (x_0, x_{01}, x_1) tel que x_{01} soit situé "entre" x_0 et x_1, on ait :

$$t \begin{vmatrix} \xi_0 & \eta_0 & \phi(x_0) - t \\ \xi_{01} & \eta_{01} & \phi(x_{01}) - t \\ \xi_1 & \eta_1 & \phi(x_1) - t \end{vmatrix} \leq 0$$

où $(\xi_0, \eta_0), (\xi_1, \eta_1), (\xi_{01}, \eta_{01})$ sont les coordonnées de x_0, x_1 et x_{01} dans un repère orthonormé de ce plan centré en $\bar{x}$ et orienté de telle sorte que

$$\begin{vmatrix} \xi_0 & \eta_0 \\ \xi_1 & \eta_1 \end{vmatrix} \geq 0.$$

La phrase "x_{01} est situé entre x_0 et x_1" signifie

$$\begin{vmatrix} \xi_0 & \eta_0 \\ \xi_{01} & \eta_{01} \end{vmatrix} \geq 0 \quad \text{et} \quad \begin{vmatrix} \xi_{01} & \eta_{01} \\ \xi_1 & \eta_1 \end{vmatrix} \geq 0.$$

Le deuxième résultat de ce même article d'Hartman est l'équivalence de la condition de pente bornée avec une condition de $n + 1$ points. C'est une généralisation de l'équivalence classique pour $n = 2$ entre la condition de pente bornée et la condition des trois points. Là encore, on obtient un critère pratique pour vérifier qu'une fonction vérifie la condition de pente bornée, qui est une généralisation du critère pratique précédent.

Le troisième résultat montre que si ϕ vérifie la condition de pente bornée, alors ϕ est aussi régulière que Γ. Plus précisément, lorsque Γ est de classe C^1, on peut recouvrir Γ par un nombre fini d'ouverts dans lesquels Γ peut être paramétrisé par des plongements ρ de classe C^1 définis sur un ouvert de $\mathbb{R}^{n-1}$. Alors on dit que ϕ est C^1 lorsque sa composée $\phi \circ \rho$ avec toute paramétrisation locale ρ de Γ est de classe C^1 (sur l'ouvert de $\mathbb{R}^{n-1}$ où est définie la paramétrisation ρ). De même, on peut remplacer C^1 par $C^{1,\lambda}$, $\lambda \in (0, 1]$. En ce sens, Hartman a montré que si Γ est C^1, (respectivement $C^{1,\lambda}$) alors ϕ est C^1 (respectivement $C^{1,\lambda}$). La preuve (notamment en dimension > 2) est délicate, d'où l'intérêt d'une preuve plus simple pour ce troisième résultat, preuve donnée dans le Chapitre 1 de cette thèse. Notons en outre que ce résultat de régularité montre que seules des fonctions assez régulières peuvent vérifier la condition de pente bornée.

Dans le second article *Convex sets and the bounded slope condition* de 1968 [45], Hartman montre que l'adhérence dans $C^0(\Gamma)$ des fonctions qui vérifient la condition de pente bornée, noté $\bar{B}(\Gamma)$ est l'ensemble des fonctions continues qui sont affines sur les parties affines de Γ, ensemble noté $\Lambda(\Gamma)$. En particulier, si Ω est strictement convexe, c'est l'ensemble des fonctions continues sur Γ.

L'ensemble $\bar{B}(\Gamma)$ a des applications intéressantes en calcul des variations. On en verra un exemple pour le problème de la continuité des minimiseurs, dans le premier chapitre de cette thèse. Cet ensemble a également été utilisé par Miranda [69] pour le problème de surface minimale avec conditions de Dirichlet (Miranda obtient l'existence de solutions généralisées lorsque ϕ est dans $\bar{B}(\Gamma)$; on pourrait d'ailleurs généraliser un tel résultat à tout lagrangien strictement convexe et globalement lipschitzien).

Il y a une inclusion facile pour ce résultat d'Hartman : si une fonction vérifie une condition de pente bornée, alors c'est la restriction d'une fonction convexe et d'une fonction concave. Ainsi sur les parties affines de Γ, ϕ est à la fois convexe et concave et donc affine. Donc $\bar{B}(\Gamma) \subset \Lambda(\Gamma)$.

L'idée pour démontrer l'autre inclusion est de considérer les cônes construits à partir des fonctions $\phi^{\pm}$ (voir (4)), c'est-à-dire :

$$\{(x,z) \in \mathbb{R}^n \times \mathbb{R} : z \geq \phi^-(x)\} \quad \text{et} \quad \{(x,z) \in \mathbb{R}^n \times \mathbb{R} : z \leq \phi^+(x)\}$$

et d'écrire leur enveloppe convexe sous la forme :

$$\{(x, \phi_t(x)) : x \in \mathbb{R}^n\} \ \ , \ \{(x, \phi^t(x)) : x \in \mathbb{R}^n\}$$

avec ϕ_t et$-\phi^t$ des fonctions convexes. Ce que disait le premier résultat du premier article est que ϕ vérifie une condition de pente bornée si et seulement si ϕ, ϕ^t, ϕ_t coïncident sur Γ pour tout t assez grand. Donc il est naturel d'espérer que lorsque ϕ est dans $\bar{B}(\Gamma)$, ϕ^t et ϕ_t convergent vers ϕ lorsque t tend vers l'infini. Ce résultat est vrai et on a en fait le résultat plus fort et découplé suivant : si ϕ est concave [convexe] sur chaque partie affine de Γ, alors ϕ^t $[\phi_t]$ converge uniformément vers ϕ sur Γ (voir l'annexe du Chapitre 1).

A partir de là, il faut travailler encore un peu pour avoir le résultat de [45]. Hartman utilise en particulier que ϕ n'est pas la restriction de n'importe quelle fonction convexe ou concave, mais bien d'une fonction qui est affine sur les "rayons" partant de l'origine $\bar{x}$.

Ce que montre en particulier la preuve de l'inclusion $\bar{B}(\Gamma) \subset \Lambda(\Gamma)$, c'est que les fonctions vérifiant la condition de pente bornée sont affines sur les parties affines de Γ. Cette propriété illustre combien cette condition peut-être restrictive, par exemple sur un polygone. *A contrario*, la condition de pente bornée est d'autant moins restrictive que l'ouvert est d'"autant plus" convexe (par exemple uniformément convexe).

Comme on le verra dans la section 0.3.6, la condition de pente bornée sert à construire des barrières. Davantage, dans le cas où le lagrangien ne dépend que de κ, les fonctions affines qui apparaissent dans la condition de pente bornée sont elles-mêmes des barrières. Ces barrières ont ceci de spécifique (par rapport aux barrières construites dans la théorie des équations elliptiques, voir [36]) qu'elles n'ont besoin d'aucune hypothèse de croissance majorante sur le lagrangien (en particulier, on n'a besoin d'aucune hypothèse sur le rapport entre la plus petite valeur propre et la plus grande valeur propre de la hessienne du lagrangien par rapport à la variable κ).

On trouve des généralisations de la condition de pente bornée dans l'article de [47] et dans les articles [61], [63]. L'expression *Generalized Bounded Slope Condition* est d'ailleurs employée en des sens différents par ces auteurs. Dans [47], il s'agit véritablement d'une généralisation au sens où les fonctions affines encadrant ϕ sont remplacées par des fonctions $C^{1,1}$. C'est une hypothèse qui ne concerne que ϕ. Néanmoins, pour construire des barrières, les auteurs recquièrent une borne sur le rapport entre la plus petite valeur propre et la plus grande valeur propre de la hessienne du lagrangien. L'expression 'Generalized Bounded Slope Condition' est moins claire dans les articles de Mariconda et Treu, où elle correspond à l'existence de sous/sur-solutions qui encadrent ϕ sur la frontière. Or le fait d'être une sous/sur-solution pour une fonction est évidemment lié au lagrangien du problème variationnel considéré. Autrement dit, chez ces derniers auteurs, la condition de pente bornée généralisée peut être satisfaite par une fonction ϕ pour certains lagrangiens et pas pour d'autres.

0.3.4 Lagrangiens généraux

Dans la plupart des articles autour du théorème de Hilbert-Haar, on étudie des problèmes variationnels où le lagrangien ne dépend que la variable κ. Il existe cependant une exception notable : l'article de Stampacchia [84]. La deuxième partie de cet article est consacrée à la généralisation de théorème de Hilbert-Haar pour des lagrangiens de la forme $L(x, z, \kappa) = G(x, z) + F(\kappa)$. La fonction F est C^2 et vérifie la condition

$$\langle \nabla^2 F(\kappa)\xi, \xi\rangle \geq \mu(1 + |\kappa|^2)^\tau |\xi|^2, \quad \mu > 0, \quad -1/2 < \tau.$$

La fonction G est continue sur $\Omega \times \mathbb{R}$, dérivable par rapport à z et G_z est continue sur $\Omega \times \mathbb{R}$. Elle vérifie la condition suivante de croissance uniformément par rapport à x :

$$\liminf_{|u|\to\infty} \frac{uG_u(x, u)}{|u|^{2\tau+2}} > -\tilde{\mu}\Lambda(2\tau + 2, \Omega), \tag{5}$$

où $\Lambda(\alpha, \Omega)$ est défini par

$$\Lambda(\alpha, \Omega) := \inf_{u \in W_0^{1,\alpha}(\Omega)} \frac{\int_\Omega |\nabla u|^\alpha}{\int_\Omega |u|^\alpha}$$

et

$$\tilde{\mu} = \frac{4^\tau \mu}{(2\tau + 2)\max(1, 2\tau + 1)}.$$

(En fait, Stampacchia prend $\tilde{\mu} = \mu$, mais son argument nous échappe dans ce cas). L'ouvert Ω est supposé de classe C^2 et uniformément convexe (voir (3)). On suppose enfin que la fonction ϕ vérifie la condition de pente bornée et qu'elle est la trace d'une fonction $W^{2,p}(\Omega), \quad p > n$.

Alors le théorème de Stampacchia affirme qu'il existe un unique minimiseur pour I dans l'ensemble des fonctions lipshitziennes qui valent ϕ au bord.

Pour montrer la nécessité des hypothèses sur G, Stampacchia propose le contrexemple suivant $L(x, z, \kappa) = |\kappa|^2 - qz^2 + 2\psi(x)z$, avec $q = \Lambda(2, \Omega), \phi = 0$. Pour des fonctions ψ génériques, ce problème n'a pas de solution (c'est une conséquence de la théorie de Fredholm appliquée à l'équation d'Euler du problème). Notons que ceci est un contrexemple au théorème d'existence. Il resterait à trouver un éventuel contrexemple au problème de la régularité : s'il existe un minimiseur dans l'ensemble des fonctions de type Sobolev, peut-il ne pas être lipschitzien si G ne vérifie pas les conditions ci-dessus ? Il s'agit d'un problème ouvert.

Dans l'article de Hartman et Stampacchia [47], des idées analogues à celles de [84] sont utilisées pour établir des estimations *a priori* sur des solutions d'équations elliptiques. L'originalité de [47] consiste à réemployer l'idée remise à jour par Miranda concernant un principe du maximum sur le gradient. L'article [47] a fortement inspiré certaines idées des Chapitres 2 et 3 pour adapter les résultats de [28] au cas d'un lagrangien plus général.

0.3.5 Théorème d'existence et de régularité simultanées

La méthode de Hilbert-Haar est à la fois un théorème d'existence et de régularité, puisqu'on cherche un minimiseur directement dans un ensemble de fonctions lipschitziennes. Il existe essentiellement deux méthodes pour obtenir l'existence d'un minimiseur dans le cadre de cette théorie.

La première est construite à partir de la méthode directe. Comme l'ensemble des fonctions lipschitziennes n'a pas de bonnes propriétés de compacité, on montre l'existence d'un minimiseur dans l'ensemble des fonctions lipschitziennes de constante $\leq K$. Ensuite des estimations a priori (voir la section suivante) permettent d'affirmer qu'il existe $Q > 0$ tel que si u_K minimise I sur l'ensemble des fonctions K lipschitziennes, alors la constante de Lipschitz de u_K est $\leq Q$. On prend alors $K > Q$. Ainsi, si v est une fonction lipschitzienne, alors $u_K + t(v - u_K)$ est une fonction K lipschitzienne pour tout t assez petit, ce qui implique

$$I(u_K) \leq I(u_K + t(v - u_K)) \leq (1 - t)I(u_K) + tI(v)$$

(par convexité de I) et finalement $I(u_K) \leq I(v)$. On en déduit que u_K est un minimiseur sur l'ensemble des fonctions lipschitziennes. Ce type de preuve apparaît dans l'article de Miranda [69] puis est repris dans [47]. (La preuve de Stampacchia [84] qui emploie la méthode directe dans un espace de Sobolev et la théorie de la régularité de De Giorgi, est néanmoins dans le même esprit).

La seconde méthode d'existence s'appuie sur l'utilisation des théorèmes généraux d'existence pour des équations elliptiques non linéaires, souvent fondés sur la théorie du degré, et auxquels ont contribué Minty, Browder, Lions... (pour des références, voir [71] et [47]). Appliquant ces théorèmes aux équations d'Euler-Lagrange, on peut obtenir l'existence de minimiseurs (voir [36], [71] et [47]).

0.3.6 Estimations a priori

Quelle que soit la stratégie adoptée par chacun des auteurs cités, ceux-ci sont tous amenés à établir des estimations *a priori* (au sens où on suppose *a priori* l'existence d'un minimiseur) sur le gradient d'un minimiseur. Comme dans la théorie de Schauder pour les équations elliptiques quasilinéaires (voir [36]), on procède en trois temps. On établit d'abord un principe de comparaison pour le minimiseur. Un tel principe affirme que si le minimiseur est encadré sur Γ entre deux fonctions vérifiant certaines propriétés (typiquement, être des sous/sur-minimiseurs, voir [37] ou la section 0.3.6.1), alors il est encadré par ces deux fonctions sur Ω. Ensuite, on construit des barrières qui encadrent ledit minimiseur sur Γ et qui vérifient lesdites propriétés. Par le principe de comparaison, elles l'encadreront aussi sur Ω. Si l'on sait borner uniformément le gradient de ces barrières, alors on en déduit une borne du gradient du minimiseur sur Γ (ce type d'argument apparaît déjà dans [46]). Enfin, on établit un principe du maximum, assertant que si u est un minimiseur, alors $||\nabla u||_{L^\infty(\Omega)} \leq ||\nabla u||_{L^\infty(\Gamma)}$. On en déduit finalement une borne sur $||\nabla u||_{L^\infty(\Omega)}$. Ecrit ainsi, cela suppose par exemple que $u \in C^1(\bar{\Omega})$. C'est ce qui a amené Stampacchia [84] ou Morrey [71] notamment a utilisé des théorèmes de régularité issus de la théorie de De Giorgi pour pouvoir manipuler des fonctions $C^1(\bar{\Omega})$ dans leurs estimations *a priori*. En écrivant différemment le principe du maximum sur le gradient, Miranda a pu s'affranchir de cette théorie pour démontrer le théorème de Hilbert-Haar.

0.3.6.1 Principe de comparaison

Dans les principes de comparaison interviennent souvent des sous/sur minimiseurs. On dit que v est un sur-minimiseur pour I si $I(v) \leq I(w)$ pour tout $w \geq v$ (et symétriquement pour un sous-minimiseur).

Pour comparer un minimiseur u avec un autre minimiseur v (voire un sous-minimiseur u et un sur-minimiseur v) en sachant que $u \leq v$ sur Γ, on écrit le plus souvent $I(u) \leq I(\min(u,v))$, voire $I(u) \leq I((1-t)u + t\min(u,v))$ pour $t \in (0,1)$ puis on exploite les hypothèses données sur L (ce qui peut s'avérer techniquement très acrobatique, comme dans [84] et [47] dont on s'est inspiré pour les principes de comparaison dans le Chapitre 2). L'intérêt des fonctions $\min(u,v)$ ou $(1-t)u + t\min(u,v)$, c'est qu'elles sont dans le même espace de Sobolev que u ou v, ou encore qu'elles ont des constantes de Lipschitz inférieures à celles de u ou v.

Notons qu'une nouvelle génération de principes de comparaison est apparue dans les articles de [24], [62], [61]. Ils sont valables dans les espaces de Sobolev et pour des lagrangiens éventuellement à valeurs $+\infty$ qui sont convexes mais pas nécessairement strictement convexes.

0.3.6.2 Barrières

De manière générale, étant donné un problème variationnel avec une condition de Dirichlet ϕ sur Γ, une barrière supérieure en un point y de Γ est une fonction $l_y^+ : \bar{\Omega} \to \mathbb{R}$, égale à ϕ en y et qui majore les éventuelles solutions du problème variationnel. On peut de même définir la notion de barrière inférieure. En pratique, pour obtenir une barrière supérieure, on construit une fonction l_y^+ qui est un sur-minimiseur du problème et qui majore ϕ sur Γ. En particulier, les sur-solutions de l'équation d'Euler-Lagrange sont des sur-minimiseurs (voir [62]).

Il est facile de voir que dans le cas où $L(x,u,\kappa)$ ne dépend que de κ, alors les constantes et les fonctions affines sont des minimiseurs (ceci n'est plus nécessairement vrai lorsque L dépend de x et u ou lorsque L est à valeurs $+\infty$). Ainsi, lorsque $L(x,u,\kappa) = F(\kappa)$, si un minimiseur (sur un ensemble de Sobolev) est borné à la frontière, il est nécessairement borné sur l'ouvert (pour des Lagrangiens plus généraux, la question est délicate : voir [84]). De même, les fonctions affines données par une condition de pente bornée vérifiée par ϕ sont des minimiseurs et elles constituent donc des barrières appropriées. Lorsque L est de la forme $L(x,u,\kappa) = F(\kappa) + G(x,u)$, Stampacchia [84] et Hartman et Stampacchia [47] ont construit des barrières à partir de ces fonctions affines, mais qui n'exigent pas d'hypothèses de croissance majorante sur le lagrangien. Une telle barrière (qui semble inédite) apparaît dans le Chapitre 2.

0.3.6.3 Principe du maximum sur le gradient

La théorie des équations elliptiques offrent divers principes du maximum sous divers types d'hypothèses (voir [84], [47] et [36]). Pour établir un principe du maximum sur le gradient, on peut montrer que les composantes du gradient vérifient une équation elliptique linéaire où ces principes s'appliquent (voir par exemple [71]). Plus pertinent pour cette thèse est le principe du maximum sur le gradient sous la formulation employée par Miranda (que ce-dernier attribue à Von Neumann). Ainsi, si u est un minimiseur (cas $L(x,u,\kappa) = F(\kappa)$), on a :

$$\max_{x\in\Omega, y\in\Gamma} \frac{|u(x)-u(y)|}{|x-y|} = \max_{x\in\Omega, y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|}.$$

Cette formulation ne fait pas intervenir le gradient de u, et peut-être établie pour des minimiseurs sur des espaces de Sobolev (dont les gradients ne sont définis que presque partout et ont un sens faible).

Ce principe du maximum découle de l'application du principe de comparaison entre un minimiseur u et son translaté $u_\tau := u(\cdot - \tau)$. Les fonctions u et u_τ sont deux minimiseurs de I sur $\Omega \cap \Omega_\tau$ où $\Omega_\tau := \Omega + \tau$. On obtient une estimation du type : pour tout $\tau > 0$,

$$\max_{x \in \Omega \cap \Omega_\tau} (u(x) - u(x - \tau)) \leq \max_{y \in \partial\Omega \cap \Omega_\tau} (u(y) - u(y - \tau)).$$

Un tel principe du maximum sera réécrit avec la même preuve par Mariconda et Treu [62] pour des minimiseurs dans un espace de fonctions de Sobolev.

0.3.7 L'idée nouvelle de Clarke

Par le principe de comparaison de Mariconda et Treu (qui est l'adaptation au cadre Sobolev du principe de comparaison classique), Clarke [28] compare u à $u_\lambda(x) := u((x - z)/\lambda + z)$ sur

$$\Omega_\lambda := \lambda(\Omega - z) + z$$

où $z \in \Omega$. Autrement dit, on remplace l'idée de comparer u et une version translatée de u par une comparaison entre u et une version dilatée de u. C'est possible parce que u_λ est un minimiseur de I sur Ω_λ.

Pour comprendre l'intérêt de cette idée, il faut revenir au principe du maximum qui découlait de la comparaison de u et d'une version translatée de u. Pour exploiter ce principe du maximum, on doit estimer le quotient :

$$\max_{x \in \Omega, y \in \Gamma} \frac{|u(x) - u(y)|}{|x - y|}.$$

Les valeurs absolues imposent d'avoir une majoration *et* une minoration de $(u(x) - u(y))$ lorsque $x \in \Omega$ et $y \in \Gamma$. Une autre manière, équivalente, d'aborder ce point est de remarquer qu'on compare u et sa version translatée u_τ sur l'ouvert $\Omega \cap \Omega_\tau$. Pour appliquer le principe de comparaison, on doit les comparer sur la frontière de cet ouvert. Or, la frontière de cet ouvert est composée d'un "morceau" de Γ et d'un "morceau" de $\partial\Omega_\tau = \Gamma + \tau$. C'est précisément la présence de ces deux morceaux qui imposent d'avoir une double inégalité dans la définition de la condition de pente bornée.

Au contraire, la frontière de $\Omega \cap \Omega_\lambda$ est composée d'un seul morceau (lorsque Ω est convexe), et c'est ce qui permet de supprimer l'une des deux inégalités dans la définition de la condition de pente bornée. Clarke définit la condition de pente minorée (ou majorée, on obtient des résultats identiques) : on dit que ϕ vérifie la condition de pente minorée de constante $Q > 0$ si pour tout $y \in \Gamma$, il existe $\zeta \in \mathbb{R}^n$, $|\zeta| \leq Q$ tel que pour tout $x \in \Gamma$,

$$\phi(y) + \langle \zeta, x - y \rangle \leq \phi(x).$$

Cette condition est nettement moins restrictive que la condition de pente bornée. Elle est étudiée dans le Chapitre 1.

Le théorème de Clarke [28] affirme que si F est strictement convexe et si ϕ vérifie la condition de pente minorée, alors tout minimiseur sur $W_0^{1,1}(\Omega) + \phi$ est localement lipschitzien.

Pour la première fois dans les résultats de la théorie Hilbert-Haar, des hypothèses globales entraînent des conclusions locales. En fait, les conclusions ne sont pas réellement locales puisque le caractère localement lipschitzien découle d'une inégalité beaucoup plus forte : on a pour tout $x, y \in \Omega$:

$$u(y) \geq u(x) - K\frac{|x-y|}{d_\Gamma(y|x)}.$$

Ici, K dépend de ϕ et de Ω et $d_\Gamma(y|x)$ est la distance de y à Γ en passant par x, c'est-à-dire égale à $|y-z|$ où $z \in \Gamma$ est le point d'intersection de Γ et de la demi-droite $[y, x)$.

Si on suppose de plus que F est coercive, Clarke obtient, par une méthode nouvelle, que u est continu sur $\bar{\Omega}$ au sens de Hölder. C'est un problème ouvert que de savoir si sans coercivité du lagrangien et sans stricte convexité de l'ouvert un minimiseur dans $W^{1,1}(\Omega)$ est continu sur $\bar{\Omega}$ (en supposant Ω convexe et ϕ vérifiant une condition de pente minorée).

0.3.8 Les contributions de cette thèse dans la théorie de Hilbert-Haar

Les trois premiers chapitres de cette thèse sont consacrés à des développements des idées apparaissant dans [28]. Dans le premier chapitre (qui correspond à [12]), j'étudie les propriétés de la condition de pente minorée. Le deuxième chapitre (article en collaboration avec Francis Clarke, voir [13]) correspond à une extension des résultats de [28] à des lagrangiens plus généraux. Dans le Chapitre 3 (correspondant à [11]), je retranscris ces résultats dans le cadre des équations elliptiques.

0.3.8.1 La condition de pente minorée

Le Chapitre 1 poursuit deux objectifs. Le premier est de comprendre dans quelle mesure la condition de pente minorée est moins restrictive que la condition de pente bornée. Le second objectif est de généraliser les résultats obtenus par Hartman dans [43], [45] au cas de la condition de pente minorée. Même si la condition de pente minorée n'impose pas à l'ouvert Ω d'être convexe, c'est pour des ouverts convexes que j'ai établi mes résultats (dont certains restent vrais pour des ouverts généraux). Il est à noter que dans les articles [28] et [13], la condition de pente minorée est utilisée seulement pour des ouverts convexes.

Il apparaît que la semiconvexité joue un rôle fondamental dans l'étude de la condition de pente minorée. On peut dire que la semiconvexité est à la condition de pente minorée ce que les fonctions $C^{1,\alpha}$ sont à la condition de pente bornée. Plus précisément, une fonction ϕ vérifie la condition de pente minorée si et seulement si c'est la restriction à Γ d'une fonction convexe, voire la restriction d'une fonction linéairement semiconvexe dans le cas d'un ouvert uniformément convexe. De plus, de la même manière qu'on a défini la propriété pour ϕ d'être de classe $C^{1,1}$ (voir section 0.3.3), on peut dire que ϕ est semiconvexe si sa composée avec toute paramétrisation locale de Γ est semiconvexe (sur un certain ouvert de $\mathbb{R}^{n-1}$).

Dans le premier chapitre, nous montrons

Théorème 0.1
Soit Ω un ouvert convexe borné, avec $\Gamma := \partial\Omega$ de classe $C^{1,1}$ et ϕ une fonction définie sur Γ. Si ϕ vérifie la condition de pente minorée, alors ϕ est linéairement semiconvexe. Si de plus Ω est uniformément convexe, alors la réciproque est vraie : si ϕ est linéairement semiconvexe, alors ϕ vérifie la condition de pente minorée.

Ce résultat correspond à certains résultats déjà cités d'Hartman pour la condition de pente bornée.

Ce résultat repose sur des propriétés des fonctions semiconvexes établies dans [23]. On a également trouvé éclairant d'introduire une notion de sous-différentiel pour ces fonctions ϕ définies sur Γ. Cette notion n'est pas nouvelle : elle consiste simplement à prolonger ϕ par $+\infty$ hors de Γ (un procédé courant en analyse non lisse) et à considérer les sous-différentiels de ces prolongements. Enfin, par les mêmes méthodes, nous donnons une preuve particulièrement simple du théorème principal d'Hartman dans [43].

Par ailleurs, figurent dans ce chapitre et son annexe des généralisations au cas de la pente minorée de résultats prouvés par Hartman pour la pente bornée. Ainsi, on vérifie que l'adhérence de l'ensemble des fonctions vérifiant la condition de pente minorée dans $C^0(\Gamma)$ est l'ensemble des fonctions de $C^0(\Gamma)$ qui sont convexes sur les parties affines de Γ. Ce résultat correspond au résultat d'Hartman rappelé dans la section 0.3.3, où les mots "minorée" et "convexes" sont remplacés respectivement par "bornée" et "affines". On a aussi introduit dans ce chapitre une application de ce résultat d'Hartman à la question de la continuité des solutions d'un problème de calcul des variations. Cette application renvoie à la question centrale de la théorie Hilbert-Haar : quelle régularité sur la condition au bord induit quelle régularité sur la solution à l'intérieur de l'ouvert ? En particulier, lorsque ϕ est continue, un minimiseur est-il nécessairement continu ? La réponse est ouverte en général (lorsque l'ouvert est supposé seulement lipschitzien) mais vraie lorsque l'ouvert et le lagrangien $L(x,u,\kappa) = F(\kappa)$ sont strictement convexes. Ainsi on a :

Théorème 0.2
Soit Ω un ouvert convexe borné de $\mathbb{R}^n$, $\phi : \Gamma \to \mathbb{R}$ une fonction continue, affine sur les parties affines de Γ et $I(u) := \int_\Omega F(\nabla u)$. Ici, F est une fonction strictement convexe sur $\mathbb{R}^n$. On considère le problème de minimiser I sur l'ensemble des fonctions $u \in W^{1,1}(\Omega)$ qui valent ϕ au bord. Si u est une solution, alors elle est continue sur $\bar{\Omega}$.

Il est peu vraisemblable que ce résultat soit nouveau. Pourtant, nous ne l'avons jamais vu publié.

Dans le Chapitre 1, on donne également un critère inspiré de [43] pour identifier les fonctions vérifiant la condition de pente minorée et on l'utilise pour montrer qu'une certaine fonction ϕ (définie ci-dessous) vérifie la condition de pente minorée mais pas la condition de pente bornée. Cet exemple est particulièrement important pour montrer l'intérêt de la condition de pente minorée en calcul des variations.

Considérons le cas (le plus simple pour les équations elliptiques) des fonctions harmoniques dans le disque unité de $\mathbb{R}^2$, sous une condition de Dirichlet sur le cercle unité. On sait que si ϕ est continue, alors il existe une unique fonction harmonique continue sur $\bar{\Omega}$ solution du problème. Il est bien connu aussi que si ϕ est dans un espace $C^{2,\alpha}$ ou $W^{2,p}$, alors la solution sera dans le même espace. Mais que dire si ϕ est supposée seulement lipschitzienne ? L'exemple suivant, qui nous a été donné par P.Cannarsa, montre que la solution n'est *pas* nécessairement lipschitzienne : la fonction

$$u(r\cos\theta, r\sin\theta) := -\sum_{i\geq 1} \frac{r^i \cos i\theta}{i^2}$$

est harmonique, et sa restriction au bord ϕ est même C^1 par morceaux (avec une seule singularité en $(1,0)$). Mais u n'est pas globalement lipschitzienne. Il n'est pas difficile de voir que $u \in W^{1,2}(\Omega)$ et donc u minimise I avec $L(x,u,\kappa) = |\kappa|^2$. On peut vérifier également que

$$\phi(\cos\theta, \sin\theta) = -\frac{\pi^2}{6} + \frac{\pi}{2}\theta - \frac{\theta^2}{4} \quad , \quad \theta \in [0, 2\pi[.$$

Dans le premier chapitre, on montre que cette fonction ϕ ne satisfait pas la condition de pente bornée, mais satisfait la condition de pente minorée.

0.3.8.2 Lagrangiens plus généraux

Dans le deuxième chapitre, on s'intéresse à la généralisation des résultats de Clarke [28] au cas de lagrangiens plus généraux. En suivant quelques techniques de [84] et [47], on a pu obtenir une telle généralisation pour des lagrangiens de la forme $L(x,u,\kappa) = F(\kappa) + G(x,u)$ où F est uniformément convexe et G localement lipschitzienne en u. Plus précisément, on introduit les hypothèses suivantes :

$(H\Omega)$ Ω est un ouvert convexe borné.

(HF) Il existe $\mu > 0$ tel que pour tout $\theta \in (0,1)$ et $\kappa, \kappa' \in \mathbb{R}^n$, on ait :

$$\theta F(\kappa) + (1-\theta)F(\kappa') \geq F(\theta\kappa + (1-\theta)\kappa') + (\mu/2)\theta(1-\theta)|\kappa - \kappa'|^2. \quad (6)$$

(HG) $G(x,u)$ est mesurable en x et différentiable en u, et pour tout intervalle borné U, il existe une constante M telle que pour presque tout $x \in \Omega$,

$$|G(x,u) - G(x,u')| \leq M|u - u'| \quad \forall u, u' \in U. \quad (7)$$

On suppose de plus qu'il existe une fonction bornée b tel que l'intégrale

$$\int_\Omega G(x, b(x))\, dx$$

soit bien définie et finie.

Sous les hypothèses $(H\Omega), (HF)$ et (HG), la fonctionnelle

$$I(w) := \int_\Omega \{F(Dw(x)) + G(x, w(x))\}\, dx$$

est bien définie sur l'ensemble des fonctions de $W^{1,1}(\Omega)$ qui sont bornées sur Ω. On appelle (P) le problème de minimiser I sur $W_0^{1,1}(\Omega) + \phi$. On conviendra de dire que u est solution de (P) relativement à $L^\infty(\Omega)$ si u est lui-même borné et si on a $I(u) \leq I(w)$ pour toute fonction w admissible pour (P) et bornée sur Ω.

Alors on a :

Théorème 0.3
Sous les hypothèses $(H\Omega), (HF)$ et (HG), et si ϕ vérifie la condition de pente minorée, toute solution u de (P) relativement à $L^\infty(\Omega)$ est localement lipschitzienne sur Ω.

Stampacchia [84] a décrit des conditions structurelles sur G qui garantissent *a priori* que les solutions de (P) (sur $W^{1,1}(\Omega)$) soient bornées.

Ce résultat est assez proche de celui figurant dans l'article de Stampacchia [84], qui exigeait cependant des hypothèses de régularité plus contraignantes sur les données, et qui demandait à ϕ de vérifier la condition de pente bornée.

Notons que dans tous les travaux s'inscrivant dans la théorie Hilbert-Haar, les seuls lagrangiens envisagés sont de cette forme (aussi bien chez Stampacchia que dans les travaux plus récents de Mariconda et Treu). En particulier, aucun résultat n'est connu pour des lagrangiens de la forme $F(x, u)$ (des théorèmes pour de tels lagrangiens pourraient laisser espérer des résultats du type Hilbert-Haar dans le cas où les fonctions admissibles sont à valeurs vectorielles, en travaillant composante par composante).

Pour généraliser le Théorème 0.3, on introduit de nouveau une version dilatée u_λ de u sur Ω_λ, idée déjà présente dans [28] comme on l'a vu. Plusieurs difficultés apparaissent : on a déjà signalé que les fonctions constantes ou affines ne sont plus des minimiseurs. Il s'agit donc de construire des barrières plus compliquées pour minorer *a priori* tout minimiseur du problème. De plus, pour éviter de faire des hypothèses de croissance majorante sur F (et rester ainsi dans l'esprit de la théorie de Hilbert-Haar), on a construit des barrières qui n'exigent que de l'uniforme convexité pour F (qui est une hypothèse de croissance minorante). Ensuite, u_λ n'est plus nécessairement un minimiseur sur Ω_λ (vu les hypothèses très générales faites sur G). Il faut donc modifier u_λ pour pouvoir le comparer avec u.

Notons que dans les articles de Mariconda et Treu, des principes de comparaison apparaissent pour des lagrangiens de la forme $L(x, u, \kappa) = F(\kappa) + G(x, u)$. L'un d'eux est même utilisé dans [62] pour comparer u et sa version translatée u_τ et obtenir ainsi un principe du maximum sur le gradient, mais alors G ne dépend pas de x et est supposé convexe, deux hypothèses très restrictives. Au contraire, nous ne supposons aucune hypothèse de monotonie ou de convexité sur G. Par ailleurs, pour établir des résultats d'existence et de régularité, Mariconda et Treu ou bien supposent *a priori* l'existence de barrières (comme dans [61]), ou bien construisent des barrières qui lient la condition au bord ϕ et le lagrangien du problème [63] (et donc comme on l'a déjà remarqué, pour une condition au bord fixée, l'existence de ces barrières n'est pas assurée pour tous les lagrangiens).

Les résultats de continuité à la frontière établis par Clarke dans [28] demeurent vrais pour des lagrangiens plus généraux, sans qu'il soit nécessaire de modifier la preuve. Dans la deuxième annexe au Chapitre 2, on a proposé d'autres généralisations : que peut-on dire lorsque l'ouvert Ω n'est pas convexe ? et lorsque F n'est plus supposé uniformément convexe ? Les réponses apportées à ces questions demeurent très limitées.

0.3.8.3 Equations elliptiques non linéaires

Le Chapitre 3 est consacré à la transcription des idées du Chapitre 2 au contexte des équations elliptiques à forme divergentielle, telles qu'elles sont traitées dans

[47]. Plus précisément, on s'intéresse aux solutions du problème

$$\int_\Omega \{\langle a(\nabla u(x)), \nabla\eta(x)\rangle - \mathcal{F}[u](x)\eta(x)\}\, dx = 0, \tag{8}$$

pour tout $\eta \in C_c^\infty(\Omega)$. Le champ de vecteurs a est supposé uniformément elliptique au sens où il existe $\nu > 0$ tel que $\langle a(\kappa) - a(\kappa'), \kappa - \kappa'\rangle \geq \nu|\kappa - \kappa'|^2$, pour tout $\kappa, \kappa' \in \mathbb{R}^n$. La fonctionnelle $\mathcal{F}$ doit vérifier certaines hypothèses de croissance. On suppose que $\mathcal{F}[u] \in L^1(\Omega)$ pour tout $u \in W^{1,2}(\Omega)$ et que

$$\mathcal{F}[u](x)\mathrm{sgn}\, u(x) \leq \sum_{i=1}^m c_i ||u||^{\beta(i)}_{L^{\alpha(i)}} |u(x)|^{\gamma(i)-1} \quad \text{p.p.} x \in \Omega \tag{9}$$

où $c_i \geq 0, \alpha(i) \geq 1, \beta(i) \geq 0, \gamma(i) \geq 1$, et $\alpha(i) \leq 2^*, \beta(i) + \gamma(i) \leq 2$. On a noté $1/2^* = 1/2 - 1/n$. On suppose aussi que les coefficients c_i dans (9) vérifient

$$\nu \sum {}' c_i \Lambda^{-2} |\Omega|^{1-2/\sigma+\beta(i)/\alpha(i)} > 0 \tag{10}$$

où $\sum'$ est la somme sur les indices i tels que $\beta(i) + \gamma(i) = 2$. Ici, $\sigma := \max(\alpha(i), 2) \leq 2^*$ et

$$\Lambda := \inf_{W_0^{1,2}(\Omega)} \frac{||\nabla u||_{L^2}}{||u||_{L^\sigma}}.$$

On suppose également les propriétés suivantes sur $\mathcal{F}$: d'abord, pour tout $M > 0$, il existe $\chi(M)$ tel que si $u \in W^{1,2}(\Omega)$ vérifie $|u(x)| \leq M$ sur Ω, alors

$$|\mathcal{F}[u](x)| \leq \chi(M). \tag{11}$$

Deuxièmement, si u_h est une suite de fonctions lipschitziennes valant ϕ au bord, bornée dans $L^\infty(\Omega)$ et convergeant uniformément sur tout compact de Ω vers u quand $h \to \infty$, alors $\mathcal{F}[u_h](x) \to \mathcal{F}[u_0](x)$ presque partout sur Ω quand $h \to \infty$.

Un exemple de fonctionnelle $\mathcal{F}[u]$ est donné par

$$\mathcal{F}[u](x) := G[u]g(x, u(x))$$

où

$$G[u] := [\int_\Omega h(x, u(x))\, dx]^{\beta-1}\ ,\ g(x,u) := \beta h_u(x,u).$$

où h est une fonction C^1. On peut voir facilement que dans ce cas, (8) est l'équation d'Euler-Lagrange du problème variationnel :

$$\min\{\int_\Omega F(\nabla u) - [\int_\Omega h(x,u)]^\beta\}.$$

Théorème 0.4
Sous les hypothèses précédentes sur le lagrangien, si Ω est un ouvert borné convexe et si ϕ vérifie une condition de pente minorée, alors il existe $u \in (W_0^{1,2}(\Omega) + \phi) \cap L^\infty(\Omega)$ solution de (8). De plus, u est localement lipschitzien sur Ω.

Ce théorème généralise certains résultats de [47], dans le sens où la condition de pente bornée est remplacée par la condition de pente minorée.

La principale originalité par rapport au Chapitre 2 consiste dans la preuve de l'existence de la solution. On s'inspire pour cela de [47], qui consiste à faire intervenir la notion de K quasi-solution. On dit que u est une K quasi-solution si

$$\int_\Omega \{\langle a(\nabla u), \nabla(v-u)\rangle - F[u](v-u)\} \geq 0,$$

pour toute fonction K lipschitzienne v qui vaut ϕ au bord. On peut montrer l'existence de K quasi-solutions par un théorème d'existence abstrait (en l'occurence, celui démontré dans la partie I de [47]). La seconde étape consiste à obtenir une estimation *a priori* sur le gradient des K quasi-solutions. L'obtention des estimations *a priori* est très similaire au travail effectué dans le Chapitre 2. On peut alors faire tendre $K \to +\infty$. La difficulté dans ce dernier procédé tient au fait que les estimations *a priori* ne sont valables que sur les compacts de Ω.

0.4 Calcul des Variations entre variétés

Comme indiqué dans la première partie de cette introduction, les trois premiers chapitres de cette thèse sont consacrés à des problèmes de calcul des variations lorsque l'ensemble des fonctions admissibles est constitué d'applications définies sur un ouvert Ω de $\mathbb{R}^n$ à valeurs dans $\mathbb{R}$.

Les deux chapitres suivants s'inscrivent dans un faisceau de questions qui émanent de problèmes de calcul des variations où l'ensemble des fonctions admissibles est constitué d'applications définies sur une variété M à valeurs dans une variété N. Dans [17], Brezis propose quatre exemples éclairants pour mettre en évidence quelques traits spécifiques à de tels problèmes. Il ressort notamment que les minimiseurs ont en général des singularités (contrairement au cas scalaire traité dans les trois premiers chapitres). Ce fait apparaît même pour des applications harmoniques (voir [79], [80]). Ces singularités peuvent être causées par des obstructions topologiques. C'est notamment le cas lorsque M est la boule unité dans $\mathbb{R}^3$, N la sphère dans $\mathbb{R}^3$ et lorsque la condition de Dirichlet $\phi : \partial M \to N$ (supposée lisse) a un degré non nul (auquel cas, il n'existe pas de fonctions continues de $\bar{M}$ dans N; on est ici confronté à un problème purement topologique, celui du prolongement de fonctions continues). De manière plus surprenante, les singularités peuvent apparaître en l'absence d'obstructions topologiques (pour un exemple, voir [17], section 1.2). Pour localiser ces singularités, un outil puisssant a été introduit dans [19] et abondamment utilisé depuis : le jacobien généralisé (voir la section 0.6.1.1).

Un autre trait frappant du calcul des variations entre variétés est le lien entre l'existence de plusieurs solutions d'une équation d'Euler-Lagrange associée à un problème variationnel et la topologie des variétés M et N. Idéalement, on cherchera un minimiseur dans chaque classe d'homotopie de l'espace des fonctions admissibles. Cela nécessite en particulier la connaissance de ces classes d'homotopie (voir la section 0.5.1.1).

L'ensemble des fonctions admissibles de ces problèmes variationnels est le plus souvent un espace de Sobolev. Il existe plusieurs définitions d'espaces de Sobolev entre variétés et ces définitions ne sont pas équivalentes. Dans toute cette thèse, on adoptera la définition suivante. Par le théorème de plongement de Whitney, on peut considérer que N est une sous-variété de $\mathbb{R}^K$. On dira alors que $f \in W^{1,p}(M,N)$ si $f \in W^{1,p}(M,\mathbb{R}^K)$ et si $f(x) \in \mathbb{R}^K$ pour presque tout $x \in M$. Donnons quelques précisions sur cette définition. La définition de $W^{1,p}(M,\mathbb{R}^K)$ se fait à partir de celle de $W^{1,p}(\mathbb{R}^m,\mathbb{R}^K)$ (où m est la dimension de M) par localisation et rectification, comme expliqué dans [71] ou encore [82] (dans toute cette thèse, M sera une variété compacte donc la localisation par partition de l'unité sera toujours possible). Enfin, si N peut être plongé dans $\mathbb{R}^K$ et dans $\mathbb{R}^L$, alors les deux espaces obtenus par la définition précédente sont homéomorphes. Pour la suite, on se contentera de considérer N comme sous-variété de $\mathbb{R}^K$, avec K fixé une fois pour toutes.

De la même manière qu'on définit $W^{1,p}(M,N)$ à partir de $W^{1,p}(\mathbb{R}^m,\mathbb{R}^K)$, on peut définir $W^{k,p}(M,N)$ à partir de $W^{k,p}(\mathbb{R}^m,\mathbb{R}^K)$ et plus généralement $W^{s,p}(M,N)$ à partir de $W^{s,p}(\mathbb{R}^m,\mathbb{R}^K)$ avec $s > 0$, $p \geq 1$. Ici, on a noté $W^{s,p}(\mathbb{R}^m,\mathbb{R}^K)$ l'ensemble des fonctions f de $W^{k,p}(\mathbb{R}^m,\mathbb{R}^K)$ (où k est la partie entière de s) telles que

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|f(x)-f(y)|^p}{|x-y|^{m+\sigma p}}\,dx\,dy < \infty \ ,$$

où $\sigma := s-k$.

Les Chapitres 4 et 5 visent à généraliser des résultats connus pour $W^{1,p}$ au cas des espaces $W^{s,p}$. L'intérêt de ces généralisations est fondé sur l'intérêt des espaces de Sobolev fractionnaires $W^{s,p}$. Ces espaces sont incontournables en théorie des traces (voir [32]) et donc pour les problèmes (de calcul des variations, d'équations aux dérivées partielles) avec conditions au bord.

0.5 Topologie de certains espaces de Sobolev

0.5.1 Quelques éléments bibliographiques

0.5.1.1 Topologie et Calcul des Variations

L'objet du Chapitre 4 de cette thèse est l'étude des classes d'homotopie des espaces de Sobolev fractionnaires $W^{s,p}(M,N)$ lorsque M et N sont deux variétés compactes et connexes, $p \geq 1$ et $s > 0$. Cette étude s'inscrit à la suite de trois articles [89], [20] et [40] qui ont apporté des contributions déterminantes pour le cas $s = 1$.

La topologie des espaces de Sobolev est une question importante pour le calcul des variations entre variétés, et plus particulièrement pour trouver plusieurs solutions à l'équation d'Euler-Lagrange associée à un problème de calcul des variations donné. Ainsi, dans un exemple de [17], on considère le système

$$-\Delta u = u|\nabla u|^2 \tag{12}$$

où les fonctions admissibles u sont définies sur le disque unité B^2 de $\mathbb{R}^2$ à valeurs dans la sphère unité S^2 de $\mathbb{R}^3$. (Ce système est donc composé de trois équations correspondant aux trois coordonnées de u).

Le système (12) est l'équation d'Euler du problème variationnel consistant à minimiser

$$I(u) := \int_{B^2} |\nabla u|^2$$

sur l'ensemble des $u \in W^{1,2}(B^2, S^2)$ sous une contrainte de type Dirichlet

$$\operatorname{tr} u|_{\partial B^2} = \phi,$$

pour une certaine fonction lisse ϕ. Dans [18], Brezis et Coron montrent que lorsque ϕ n'est pas constante, le problème (12) admet au moins deux solutions distinctes. Le coeur de la preuve consiste à montrer que l'ensemble des $u \in W^{1,2}(B^2, S^2)$ valant ϕ au bord admet un nombre infini de composantes connexes. Chacune de ces composantes connexes est caractérisée par un degré. Pour trouver des solutions à (12), l'idée est de minimiser I sur chacune des composantes connexes. (Néanmoins, I n'a pas nécessairement de minimum dans chaque composante connexe. En particulier, la méthode directe d'existence en calcul des variations est mise en échec parce que les composantes connexes ne sont pas nécessairement faiblement séquentiellement compactes, voir [18]).

Cette même stratégie pour trouver plusieurs solutions à un système qui est l'équation d'Euler d'un problème variationnel est développée dans les deux articles de White [88], [89]. Plus précisément, dans [88], le problème variationnel envisagé est de minimiser

$$I(u) := \int_M |\nabla u(x)|^p \, dx.$$

Ici, l'ensemble des fonctions admissibles n'est pas l'ensemble des fonctions de Sobolev $W^{1,p}(M, N)$ que l'on a défini précédemment mais l'adhérence dans $W^{1,p}(M, N)$ de l'ensemble des fonctions lipschitziennes de M dans N. On note dans la suite $L^{1,p}(M, N)$ cet ensemble. En notant $d := [p]$ la partie entière de p et en introduisant une triangulation de M, on définit une classe de d homotopie comme une classe d'homotopie dans l'ensemble des fonctions continues définies sur le squelette de dimension d de M (noté M^d dans la suite) à valeurs dans N. White montre qu'on peut associer à chaque élément u de $L^{1,p}(M, N)$ une classe de d homotopie. (En fait, c'est la classe de d homotopie des restrictions à M^d de l'ensemble des fonctions lipschitziennes appartenant à la composante connexe de $L^{1,p}(M, N)$ contenant u). On peut voir alors que l'infimum de I sur l'ensemble des fonctions lispchitziennes homotopes à une certaine fonction g ne dépend que de la classe de d homotopie de g. En particulier, cet infimum est nul si et seulement si g a la même classe de d homotopie qu'une constante.

Dans [89], ce n'est plus à la valeur de l'infimum mais à l'existence des minima que White s'intéresse. Il considère le cas de l'ensemble $W^{1,p}(M, N)$. Pour la valeur $d := [p] - 1$ cette fois, il associe de nouveau à chaque élément de $W^{1,p}(M, N)$ une classe de d homotopie. Il montre aussi que pour chaque application continue g sur le squelette M^{d+1} de dimension $d+1$ de M à valeurs dans N, il existe une application qui minimise I sur l'ensemble des $f \in W^{1,p}(M, N)$ qui ont la même classe de d homotopie que g.

Dans ces deux cas, on a partitionné l'ensemble des fonctions admissibles en classes (les classes de d homotopie) où la méthode directe s'applique (car ces classes de d homotopie ont les propriétés de compacité requises).

0.5.1.2 Le cas de l'espace de Sobolev $W^{1,p}$

Comme on l'a vu dans le paragraphe précédent, White s'intéresse dans [89] à la topologie de $W^{1,p}(M, N)$ en introduisant la notion de classe de d homotopie, où $d = [p-1]$. Il convient de préciser cette notion à présent. Notons h une triangulation de M définie sur un complexe simplicial K. La variété M est considérée comme une sous-variété d'un espace euclidien $\mathbb{R}^L$. Il existe $\epsilon_M > 0$ tel que la projection orthogonale Π_M sur M soit bien définie sur un voisinage tubulaire U de M d'épaisseur ϵ_M. Notons K^d le squelette de K de dimension d. Soit f un élément de $W^{1,p}(M,N)$. Alors White montre que pour presque tout $|v| < \epsilon_M$, les fonctions $g^v : K^d \to N$ définies par

$$g^v(x) := f \circ \Pi_M(h(x) + v)$$

sont continues et de plus appartiennent à la même classe d'homotopie dans $C^0(K^d, N)$. C'est cette classe d'homotopie qu'on appelle la classe de d homotopie de f. *A priori*, cette notion dépend de la triangulation h choisie. En réalité, si h_1, h_2 sont deux triangulations et que f_1, f_2 sont deux éléments de $W^{1,p}(M,N)$, si f_1, f_2 ont la même classe de d homotopie relativement à h_1, alors f_1, f_2 ont la même classe de d homotopie relativement à h_2.

La proposition 3.3 de [89] montre en particulier que les classes de d homotopie sont ouvertes (pour la métrique de $W^{1,p}(M,N)$). Ce résultat implique que si deux applications f_1, f_2 sont homotopes dans $W^{1,p}(M,N)$, alors elles ont même classe de d homotopie. La preuve fait notamment intervenir trois arguments récurrents pour ce type de questions (et notamment dans cette thèse). Le premier argument est un théorème de type Morrey adapté au cas des complexes simpliciaux. Ainsi, si K est un complexe simplicial de dimension d tel que $d < p$, alors pour tout $\epsilon > 0$, il existe $C(\epsilon) > 0$ tel que pour tout $f : K \to \mathbb{R}$ lipschitzienne, on a :

$$||f||_{L^\infty} \leq \epsilon ||Df||_{L^p} + C(\epsilon)||f||_{L^p}.$$

Le deuxième argument fait intervenir le théorème de Fubini. Par exemple, on peut affirmer que si $h : K \to M$ est une triangulation de M, alors pour tout $F \in L^1(U)$ et pour presque tout v tel que $|v| < \epsilon_M$, on a :

$$\int_{x \in K} |F(h(x) + v)|\, dx < \infty.$$

Le troisième ingrédient est constitué par des théorèmes d'extension homotopique dans le cadre des complexes simpliciaux.

Le résultat de White permettait seulement d'affirmer que deux éléments homotopes de $W^{1,p}(M,N)$ avaient le même type de d homotopie, avec $d := [p] - 1$. La réciproque a été prouvée par Hang et Lin dans [40] : si deux applications f_1, f_2 ont même classe de d homotopie, alors elles sont homotopes dans $W^{1,p}(M,N)$.

Ce résultat permet de relier l'étude des classes d'homotopie de l'ensemble $W^{1,p}(M,N)$ à une connaissance uniquement topologique de M et N. En effet, un corollaire de l'équivalence entre homotopie et d homotopie est une bijection (pour $p < \dim M$), entre

$$W^{1,p}(M,N)/\sim_{1,p} \longleftrightarrow C^0(M^{[p]}, N)/\sim_{M^{[p]-1}}.$$

Ici, $C^0(M^{[p]}, N)$ est l'ensemble des fonctions continues d'un squelette de dimension $[p]$ de M à valeurs dans N. Sur cet ensemble, on définit la relation d'équivalence

$$\forall f, g \in C^0(M^{[p]}, N), f \sim_{M^{[p]-1}} g$$

si et seulement si $f|_{M^{[p]-1}}$ et $g|_{M^{[p]-1}}$ sont homotopes dans $C^0(M^{[p]-1}, N)$. D'autre part,

$$\forall f, g \in W^{1,p}(M, N), f \sim_{1,p} g$$

signifie bien sûr que f et g sont homotopes dans $W^{1,p}(M, N)$.

On en déduit notamment que les classes d'homotopie de $W^{1,p}(M, N)$ sont en bijection avec les classes d'homotopie de $C^0(M, N)$ lorsque $p < \dim M$ et $\pi_i(N) = 0$ pour $[p] \leq i \leq \dim M$. Ceci reste vrai lorsque $p \geq \dim M$ mais la preuve est différente. Sous certaines conditions purement topologiques (faisant seulement intervenir les groupes fondamentaux de M et N), on peut aussi affirmer que $W^{1,p}(M, N)$ est connexe par arcs.

Outre les trois arguments de White décrits ci-dessus, Hang et Lin ont exploité un "outil" introduit par Brezis et Li dans [20] et que ces derniers auteurs ont baptisé "filling a hole" ("remplissage d'un trou"). L'article [20] avait été le premier à étudier systématiquement les composantes connexes de l'espace $W^{1,p}(M, N)$ en fonction des propriétés topologiques ou géométriques de M et N (le cas où M et N sont des sphères est essentiellement résolu dans cet article). Les arguments employés sont indépendants de ceux de White. La partie technique de [20] contient un certain nombre de lemmes qui montrent qu'on peut déformer continûment un élement de $W^{1,p}(M, N)$ au voisinage d'un point en un autre élément de $W^{1,p}(M, N)$ vérifiant certaines propriétés (par exemple, la propriété d'être localement constant). Parmi ces lemmes, on a déjà noté que le "filling a hole" joue un rôle important dans [40] (et c'est aussi le cas dans cette thèse). Il consiste à modifier un élément $u \in W^{1,p}(M, N)$ au voisinage d'un point x de M en un autre élément $v \in W^{1,p}(M, N)$ égal à u loin de x et qui vaut $u(rx/|x|)$ dans la petite boule de centre x et de rayon r.

Dans la seconde partie de [20], ces lemmes sont utilisés pour obtenir des résultats d'homotopie sur les ensembles $W^{1,p}(M, N)$ (l'article [40] complètera ces résultats). En particulier, Brezis et Li obtiennent que si $1 \leq p < 2 \leq \dim M$, alors $W^{1,p}(M, N)$ est connexe par arcs (ceci est un cas particulier du résultat démontré par Hang et Lin puisque si $p < 2$, on a $[p] - 1 = 0$ et deux fonctions restreintes à des points sont toujours homotopes, par connexité de N).

Une autre question introduite dans l'article [20] concerne les changements d'homotopie lorsque la valeur de p varie. Plus précisément, lorsque $q \geq p$, on peut envoyer les composantes connexes de $W^{1,q}$ sur les composantes connexes de $W^{1,p}$. Notons $i_{p,q}$ une telle application. Brezis et Li parlent de "changement d'homotopie" en p lorsque pour tout $\epsilon \in (0, p-1)$, l'application $i_{p-\epsilon,p+\epsilon}$ n'est pas bijective. Ils conjecturent également que les changements d'homotopie ont lieu seulement en des valeurs entières de p. Cette conjecture sera résolue par l'affirmative par Hang et Lin dans [40].

La généralisation de ces résultats au cas des espaces de Sobolev fractionnaires n'apparaît que dans l'article de Brezis et Mironescu [14] lorsque la variété cible est $N = S^1$ (et lorsque M est un ouvert connexe borné lisse d'un espace euclidien). Dans ce cas, si $sp < 2$, alors $W^{s,p}(M, S^1)$ est connexe par arcs tandis que si $sp \geq 2$, les classes d'homotopie de $W^{s,p}(M, S^1)$ sont en bijection avec les classes d'homotopie de $C^0(M, S^1)$.

0.5.2 Les contributions de cette thèse

0.5.2.1 Généralisation aux espaces de Sobolev fractionnaires

Dans le Chapitre 4 de cette thèse, notre objectif a été d'étendre les résultats de [89], [20], [40] et [14] aux espaces de Sobolev fractionnaires $W^{s,p}(M, N)$ lorsque $s \neq 1$ et lorsque $N \neq S^1$. La généralisation des preuves de [40] à ces espaces est assez directe, mais doit tenir compte néanmoins de deux difficultés. La première difficulté concerne le "recollement" de deux éléments de $W^{s,p}$. Ainsi, si $u_+ \in W^{s,p}(\mathbb{R}^N_+)$ et $u_- \in W^{s,p}(\mathbb{R}^N_-)$ ont la même trace sur $W^{s,p}(\mathbb{R}^{N-1} \times \{0\})$, il n'est pas vrai que la fonction u qui vaut u_+ sur $\mathbb{R}^N_+$ et u_- sur $\mathbb{R}^N_-$ soit dans $W^{s,p}(\mathbb{R}^N)$, à moins que $s < 1 + 1/p$. De plus, on ne peut parler de traces que si $sp > 1$. Lorsqu'on se place dans le cas $1/p < s < 1 + 1/p$, on rencontre une deuxième difficulté pour généraliser les arguments de Hang et Lin. Chaque fois qu'ils le peuvent, ces-derniers manipulent les fonctions lipschitziennes, qui sont continues, au lieu des fonctions des espaces de Sobolev, qui ne sont définies que presque partout. Ils peuvent le faire parce que les fonctions lipschitziennes sont contenues dans $W^{1,p}(M, N)$ et denses dans $W^{1,p}(M, \mathbb{R}^K)$. Ce n'est plus le cas des espaces $W^{s,p}(M, N)$ lorsque $s > 1$. On pourrait songer alors à remplacer l'ensemble des fonctions lipschitziennes par l'ensemble des fonctions $W^{2,\infty}$. Mais autant il est possible de recoller deux fonctions lipschitziennes qui coïncident dans l'intersection des domaines où elles sont définies, autant un tel procédé est impossible dans $W^{2,\infty}$. C'est pourquoi j'ai préféré travailler directement avec les fonctions de $W^{s,p}(M, N)$ sans passer par un ensemble intermédiaire de fonctions bénéficiant de bonnes propriétés. Cela a occasionné des modifications substantielles dans les preuves de [40].

Dans le cas $sp < 1$, la notion de trace n'est certes pas définie mais le "recollement" est toujours possible. Le cas $sp = 1$ est le plus problématique. Dans [14], ce cas était résolu grâce au concept de "good restrictions" ("bonnes restrictions"), qui est une sorte de théorème de Fubini pour les traces. Lorsqu'on ne peut définir la trace d'un élément u sur le bord Γ d'un domaine, on peut néanmoins définir la restriction de u sur presque toutes les surfaces proches de Γ. Ce concept de "good restrictions" ne semblait pas pouvoir s'insérer convenablement dans les arguments de [40]. En revanche, il était bien adapté aux différents lemmes techniques de la première partie de l'article de Brezis et Li [20]. J'ai donc généralisé ces lemmes techniques au cas des espaces $W^{s,p}(M, N)$ (pour $s < 1 + 1/p$). Le lemme "filling a hole" avait déjà été étendu au cas des espaces de Sobolev fractionnaires dans [14]. La preuve des autres lemmes de [20] reposait sur des calculs explicites dans les espaces $W^{1,p}$ et notamment sur des estimations de gradients d'éléments $u \in W^{1,p}$ en norme L^p. De telles estimations ne sont évidemment plus suffisantes lorsqu'on passe au cas des espaces de Sobolev fractionnaires puisque la quantité déterminante devient alors :

$$\int\int \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy.$$

Mais cette dernière quantité n'est pas locale. Précisons ce qu'on entend par *local* sur l'exemple suivant : on considère un élément $u \in W^{1,p}(B_2, \mathbb{R})$ (où B_2 est la boule de $\mathbb{R}^n$ centrée en 0 et de rayon 2). On modifie continûment u dans B_1 en laissant u "intact" dans $B_2 \setminus B_1$. Si on note v l'élément ainsi obtenu, on pourra estimer

$$||\nabla v||_{L^p(B_2)} \leq ||\nabla v||_{L^p(B_1)} + ||\nabla u||_{L^p(B_2 \setminus B_1)}.$$

(Bien sûr, pour que v appartienne à $W^{1,p}(B_2)$, des conditions de compatibilité des traces doivent être assurées sur la sphère unité).

En revanche, dans le cas des espaces de Sobolev fractionnaires, on sera conduit à estimer

$$\int_{B_1}\int_{B_1} \frac{|\nabla v(x) - \nabla v(y)|^p}{|x-y|^{n+sp}}\, dx\, dy + 2\int_{B_1}\int_{B_2} \ldots + \int_{B_2}\int_{B_2} \ldots .$$

Typiquement, le terme croisé $\int_{B_1}\int_{B_2} \ldots$ peut être considéré comme non local. De plus, il est souvent plus difficile d'estimer le terme fractionnaire

$$\frac{|\nabla v(x) - \nabla v(y)|^p}{|x-y|^{n+sp}}$$

que le terme $|\nabla v|^p$ (qui se calcule explicitement dans chacun des lemmes de [20]).

Pour ces différentes raisons, je n'ai pas généralisé les preuves des lemmes de [20] et j'ai préféré les démontrer par des méthodes nouvelles. Le fait remarquable est que les deux lemmes "problématiques", dénommés "bridging" et "opening a map" par Brezis et Li, peuvent être démontrés en se ramenant au lemme "filling a hole".

En regroupant les cas $sp < 1, sp = 1, sp > 1$, le résultat principal du Chapitre 4 s'énonce ainsi :

Théorème 0.5
Soit $s \in (0, 1 + 1/p), sp < \dim M$ avec $\dim M > 1$. Soient $u, v \in W^{s,p}(M, N)$. Alors u et v sont homotopes dans $W^{s,p}(M, N)$ si et seulement si u et v ont la même classe de $[sp] - 1$ homotopie.

La notion de classe de $[sp]-1$ homotopie généralise la notion analogue introduite par White [89] et redéfinie par Hang et Lin [40]. Dire que u et v ont la même classe de $[sp] - 1$ homotopie signifie que pour un squelette générique $M^{[sp]-1}$ de dimension $[sp] - 1$ de M, les restrictions de u et v à $M^{[sp]-1}$ sont homotopes (dans $C^0(M^{[sp]-1}, N)$).

Lorsque $sp \geq \dim M$, les composantes connexes de $W^{s,p}(M, N)$ sont comparables aux composantes connexes de $C^0(M, N)$ (voir l'annexe de [20]).

Le théorème 0.5, qui est la généralisation du théorème principal de [40], a de nombreux corollaires. Par exemple, lorsque $sp < 2$, l'espace $W^{s,p}(M, N)$ est connexe par arcs (ceci généralise un théorème de [20]).

Dans le cas $2 \leq sp < \dim M$ et $s \in (0, 1 + 1/p)$, lorsqu'il existe $k \leq [sp] - 1$ tel que

$$\pi_i(M) = 0\ ,\ 1 \leq i \leq k \quad \text{et} \quad \pi_i(N) = 0\ ,\ k + 1 \leq i \leq [sp] - 1,$$

alors l'espace $W^{s,p}(M, N)$ est connexe par arcs. Pour ces mêmes valeurs de s et p, les classes d'homotopie de $W^{s,p}(M, N)$ sont en bijection avec les classes d'homotopie de $C^0(M, N)$ lorsque

$$\pi_i(N) = 0\ ,\ [sp] \leq i \leq \dim M.$$

Enfin, la question des changements des classes d'homotopie trouve aussi une réponse dans le cadre des espaces de Sobolev fractionnaires. Ainsi, lorsque $W^{s_1,p_1}(M,N) \subset W^{s_2,p_2}(M,N)$, on peut définir une application naturelle qui associe à chaque classe d'homotopie de $W^{s_1,p_1}(M,N)$ une classe d'homotopie de $W^{s_2,p_2}(M,N)$. Cette application est une bijection si $[s_1p_1] = [s_2p_2]$. Autrement dit, un changement d'homotopie dans l'échelle des espaces $W^{s,p}(M,N)$ ne peut advenir que pour des valeurs non entières de $[sp]$.

0.5.2.2 Perspectives

Dans le Chapitre 4, je n'ai pas été capable de traiter le cas de $W^{s,p}(M,N)$ pour $s \geq 1 + 1/p$. La raison en est que pour "relier" continûment deux éléments u et v de $W^{s,p}(M,N)$, on déforme localement u et v en $\tilde{u}$ et $\tilde{v}$ sur un "petit ouvert" U de M sans modifier u et v hors de U (ceci est vrai aussi bien pour la méthode Brezis-Li que pour la méthode Hang-Lin). On a ensuite besoin d'affirmer que la fonction qui vaut $\tilde{u}$ sur U et u hors de U est encore dans $W^{s,p}$ (et similairement pour v). Or, la théorie des traces n'autorise cette affirmation que dans le cas $s < 1 + 1/p$.

A ma connaissance, il n'existe pas de résultats sur les composantes connexes des espaces $W^{2,p}$. On peut penser que certains outils de [20] peuvent se généraliser à ce cas. En particulier, pour modifier localement un élément u de l'espace $W^{1,p}(M,N)$, Brezis et Li introduisent souvent une famille $u^t := u \circ \rho^t$ où ρ^t a une régularité seulement lipschitzienne. Si on renforce les hypothèses de régularité sur ρ^t, (disons ρ^t de classe C^2), on peut espérer généraliser certains des lemmes de [20], voire certains théorèmes du même article (lorsque l'énoncé de ces théorèmes ne contient pas l'hypothèse $p < 2$).

Il semble difficile de généraliser les méthodes de Hang et Lin au cas des espaces $W^{2,p}(M,N)$. Toutefois, si l'on obtient des résultats pour les espaces $W^{2,p}$ (et plus généralement $W^{k,p}$), il est vraisemblable qu'on puisse les généraliser aux espaces $W^{s,p}$ avec $s \geq 1 + 1/p$.

Par ailleurs, il est raisonnable de penser que des méthodes similaires à celles employées dans le Chapitre 4 peuvent apporter des éléments de réponse à deux types de problème.

Le premier problème concerne la densité des fonctions lisses, ou des fonctions lisses sauf en un nombre fini de points, dans les espaces $W^{s,p}(M,N)$. Ce problème a été traité par Bethuel dans [5] (voir aussi [40]) pour le cas $W^{1,p}(M,N)$ et dans [6] pour le cas $W^{1-1/p,p}(\partial M,N)$.

Le deuxième problème concerne la question du prologement : si M est une variété à bord et si la fonction $g \in W^{s,p}(\partial M,N)$ est donnée, existe-t-il $u \in W^{s+1/p,p}(M,N)$ dont la trace sur ∂M est égale à g ? Le cas $s = 1 - 1/p$ a notamment été envisagé dans [42], [4] et [89], mais rien n'est connu pour d'autres valeurs de s.

0.6 Le jacobien

0.6.1 Quelques éléments bibliographiques

0.6.1.1 Un détecteur de singularités

Le jacobien généralisé auquel on s'est intéressé dans le Chapitre 5 apparaît dans un contexte variationnel dans l'article "Harmonic maps with defects " de Brezis, Coron et Lieb [19]. Etant donné un nombre fini de points $a_1, .., a_k$ de $\mathbb{R}^3$, auxquels on associe k entiers $d_1, .., d_k$, on considère l'ensemble $\mathcal{E}$ des fonctions continues ϕ sur $\mathbb{R}^3 \setminus \{a_1, .., a_k\}$ à valeurs dans S^2 telles que $\deg(\phi, a_i) = d_i$ et $\int_{\mathbb{R}^3} |\nabla\phi|^2 < \infty$. On définit $\deg(\phi, a_i)$ comme le degré de la fonction ϕ restreinte à toute petite sphère centrée en a_i (cette restriction est alors une application continue d'une sphère dans une sphère de même dimension, dont on sait définir le degré ; il est facile de voir que la valeur du degré ne dépend pas du rayon de cette petite sphère). Si la fonction ϕ est continue en a_i, alors son degré en a_i est nul. Autrement dit, la valeur du degré nous renseigne sur la "force" de la singularité topologique en a_i. Notons enfin que la condition $\int_{\mathbb{R}^3} |\nabla\phi|^2 < \infty$ implique que le degré de la fonction ϕ restreinte à des sphères centrées en 0 et de rayon suffisamment grand est nul. Cela a pour conséquence que $\mathcal{E}$ n'est non vide que si $\sum_i d_i = 0$.

Les auteurs de [19] s'intéressent alors au problème de minimiser

$$I(u) := \int_{\mathbb{R}^3} |\nabla u|^2$$

sur l'ensemble $\mathcal{E}$. En d'autres termes, on cherche l'énergie minimale lorsque le lieu et le degré topologique des singularités sont prescrits.

La solution s'exprime de manière particulièrement élégante en termes de "connexions minimales." Comme $\sum_i d_i = 0$, on peut ranger les singularités a_i en singularités positives (celles pour lesquelles $d_i > 0$) et en singularités négatives (celles pour lesquelles $d_i < 0$) (on omet les singularités de degré 0). En répétant d_i fois chaque singularité a_i, on obtient une liste de singularités positives $P_1, .., P_l$ et de singularités négatives $Q_1, .., Q_l$. La connexion minimale est alors définie par

$$L := \min \sum_{j=1}^{l} |P_j - Q_{\sigma(j)}|$$

où le minimum est pris sur l'ensemble des permutations σ de $[|1, l|]$.

On a alors le résultat

$$\inf_{\mathcal{E}} I = 8\pi L.$$

La preuve procède par double inégalité. C'est l'inégalité $\geq$ qui fait intervenir le jacobien généralisé. On définit d'abord un champ de vecteurs D associé à un élément u de $\mathcal{E}$ par :

$$D := \frac{1}{3}(\det(u, u_y, u_z), \det(u_x, u, u_z), \det(u_x, u_y, u)).$$

(La notation det désigne le déterminant pour les matrices carrées d'ordre 3 car la fonction u est vue comme un vecteur de $\mathbb{R}^3$). Il est à noter que $D \in L^1(\mathbb{R}^3, \mathbb{R}^3)$. Lorsqu'on calcule formellement $\operatorname{div} D$, on trouve le jacobien classique de u. Le jacobien généralisé de u est par définition $T(u) := \operatorname{div} D$ entendu au sens des distributions. Deux propriétés du jacobien généralisé sont mises en avant dans [19] (et utilisées pour démontrer l'inégalité $\geq$). D'une part, si $u \in \mathcal{E}$, on a

$$T(u) = \frac{4\pi}{3} \sum_{i=1}^{k} d_i \delta_{a_i}$$

(où δ_{a_i} est la somme de Dirac en a_i). D'autre part,

$$\max_{||\nabla\zeta||_{L^\infty} \leq 1} \langle T(u), \zeta \rangle = \frac{4\pi}{3} L.$$

Ce jacobien généralisé a été utilisé dans de très nombreux problèmes de calcul des variations entre variétés, non seulement pour localiser les singularités topologiques des solutions, mais également en liaison avec la notion d'énergie relaxée (l'énergie relaxée a notamment été étudiée dans l'article de Bethuel, Brezis et Coron [3], voir aussi [34]). Pour illustrer ce dernier fait, je cite ici un résultat obtenu dans le cadre $W^{1,1}(S^2, S^1)$ par Brezis, Mironescu et Ponce [22] (un résultat proche avait été prouvé auparavant dans le cadre $W^{1,2}(B^3, S^2)$, voir [3], et dans le cadre $H^{1/2}(S^2, S^1)$, voir [9]). On définit l'énergie relaxée d'un élément $u \in W^{1,1}(S^2, S^1)$ par

$$E_{\text{rel}} = \inf\{\liminf_{n\to\infty} \int_{S^2} |\nabla u_n|\}$$

où l'infimum est pris sur l'ensemble des suites d'éléments $u_n \in C^\infty(S^2, S^1)$ convergeant presque partout vers u. Comme précédemment, on peut définir le champ de vecteurs $D := \frac{1}{2}(\det(u, u_y), \det(u_x, u))$ (noter que $D \in L^1(S^2, \mathbb{R}^2)$) et le jacobien généralisé de $u \in W^{1,1}(S^2, S^1)$:

$$T(u) := \operatorname{div} D. \tag{13}$$

Alors il existe deux suites de points P_i, N_i dans S^2 telles que

$$\sum_i |P_i - N_i| < \infty \text{ et } T(u) = \pi \sum (\delta_{P_i} - \delta_{N_i})$$

(la somme doit être comprise dans le dual de $W^{1,\infty}(S^2, \mathbb{R})$). De plus, si on définit

$$L(u) := \frac{1}{\pi} \max_{||\nabla\zeta||_{L^\infty} \leq 1} \langle T(u), \zeta \rangle$$

on a $L(u) = \inf \sum_j d(\tilde{P}_j, \tilde{N}_j)$ où d est la distance géodésique sur S^2 et l'infimum est pris sur toutes les suites $(\tilde{P}_j), (\tilde{N}_j)$ qui vérifient

$$\sum (\delta_{\tilde{P}_j} - \delta_{\tilde{N}_j}) = \sum (\delta_{P_i} - \delta_{N_i})$$

(égalité comprise dans le dual de $W^{1,\infty}(S^2,\mathbb{R})$). On voit donc que les fonctions T et L généralisent au cadre $W^{1,1}$ le jacobien généralisé et la connexion minimale introduits pour les fonctions $W^{1,2}$ continues sauf en un nombre fini de points.

Notons également que la réciproque du résultat mentionné ci-dessus est vraie (voir [9]). Plus précisément, étant donné deux suites $(P_i),(N_i)$ de points sur S^2 telles que $\sum_i d(P_i,N_i)<\infty$, il existe un élément $u\in W^{1,1}(S^2,S^1)$ tel que $T(u)=\pi\sum_i(\delta_{P_i}-\delta_{N_i})$. Autrement dit, l'image du jacobien généralisé lorsqu'il est défini sur l'espace $W^{1,1}(S^2,S^1)$ est exactement l'ensemble des

$$\pi\sum_i(\delta_{P_i}-\delta_{N_i})$$

(où la somme est entendue au sens $(W^{1,\infty}(S^2,\mathbb{R}))^*$). On peut voir ([9]) que cet ensemble est égal à

$$\overline{\pi\sum_{\text{fini}}(\delta_{P_i}-\delta_{N_i})}^{(W^{1,\infty}(S^2,\mathbb{R}))^*}.$$

L'intérêt de la connexion minimale dans ce contexte est de relier énergie relaxée et énergie, par l'identité :

$$E_{\text{rel}}(u)-\int_{S^2}|\nabla u|=2\pi L(u).$$

0.6.1.2 Domaines et images du jacobien

Le jacobien généralisé a été étendu et utilisé dans de nombreux espaces fonctionnels entre variétés. Outre les espaces $W^{1,2}(\mathbb{R}^3,S^2)$ et $W^{1,1}(S^2,S^1)$ mentionnés dans le paragraphe précédent, on peut également citer le cas des fonctions à variations bornées de S^2 dans S^1 (voir [51]) et aussi le cas de la dimension supérieure : si $u\in W^{1,N-1}(S^N,S^{N-1})$, on peut définir le champ de vecteurs dans $L^1(S^N,\mathbb{R}^N)$:

$$D(u)=\frac{1}{N}(D_1,..,D_N)\quad\text{avec}\quad D_j=\det(u_{x_1},..,u_{x_{j-1}},u,u_{x_{j+1}},..,u_{x_N}). \tag{14}$$

On peut alors définir le jacobien généralisé

$$T(u):=\operatorname{div}D(u) \tag{15}$$

et sa “norme” :

$$L(u):=\frac{1}{|B^N|}\max_{||\nabla\zeta||_{L^\infty}\leq 1}\langle T(u),\zeta\rangle$$

(où $|B^N|$ est le volume de la boule unité dans $\mathbb{R}^N$). On obtient alors des résultats tout à fait similaires au cas de la dimension $N=2$ (voir [22]). Le problème abordé par Brezis, Coron et Lieb dans [19] et mentionné dans la section précédente est également généralisé à la dimension supérieure.

Plus délicate est l'extension du jacobien généralisé aux espaces fonctionnels où on ne peut plus définir simplement le champ de vecteurs $D(u)$ avec des coordonnées dans L^1. C'est pourtant l'un des résultats principaux de [10] (voir aussi [9]). Ainsi, si $N-1<p<\infty$, il existe une unique application continue

$$\bar{T}:W^{(N-1)/p,p}(S^N,S^{N-1})\to(W^{1,\infty}(S^N,\mathbb{R}))^*$$

telle que, pour tout $\zeta \in W^{1,\infty}(S^N, \mathbb{R})$, et tout $u \in W^{(N-1)/p,p}(S^N, S^{N-1})$,

$$|\langle \bar{T}(u), \zeta \rangle| \leq C_{p,N} ||u||^p_{W^{(N-1)/p,p}} ||\nabla \zeta||_{L^\infty}$$

et pour tout $u \in W^{1,N-1}(S^N, S^{N-1}) \cap W^{(N-1)/p,p}(S^N, S^{N-1})$,

$$\langle \bar{T}(u), \zeta \rangle = \langle T(u), \zeta \rangle.$$

(Ici, $T(u)$ désigne le jacobien généralisé pour les fonctions de $W^{1,N-1}(S^N, S^{N-1})$ tel qu'il a été défini en (15).)

De plus, pour tout $u \in W^{(N-1)/p,p}(S^N, S^{N-1})$, il existe des suites (P_i), (N_i) de points de S^N telles que

$$\sum_i |P_i - N_i| \leq C_p ||u||^p_{W^{(N-1)/p,p}}$$

et pour tout $\zeta \in W^{1,\infty}(S^N, \mathbb{R})$,

$$\langle T(u), \zeta \rangle = |B^N| \sum_i (\zeta(P_i) - \zeta(N_i)).$$

(où $|B^N|$ est le volume de la boule unité dans $\mathbb{R}^N$).

Dans tous les espaces fonctionnels ci-dessus, on a considéré des fonctions d'une variété de dimension N vers S^{N-1}. Il est possible d'étendre le jacobien généralisé à d'autres situations où la dimension de la variété "source" n'est pas liée à celle de la variété "but" (voir [34], [52], et [2]).

Lorsque $N > k$, le jacobien généralisé d'un élément u dans $W^{1,k-1}(S^N, S^{k-1})$ met en évidence des singularités topologiques qui ne sont plus des points, mais des objets géométriques de dimension $N - k$. Par exemple, si u est un élément de $W^{1,k-1}(S^N, S^{k-1})$ régulier hors d'une surface lipschitzienne S de dimension $N - k$ et sans frontière, alors le jacobien de u peut être vu comme un courant intégral supporté par S, et de multiplicité le degré de la fonction u restreinte à des petites sphères de dimension $k - 1$ "entourant" S (pour un énoncé précis, voir [2] et aussi [52] ainsi que le Chapitre 5).

Dans le contexte des fonctions $u \in W^{1,k-1}(S^N, S^{k-1})$, on peut obtenir des résultats sur le jacobien qui généralisent le cas $W^{1,1}(S^2, S^1)$. Ainsi, le résultat principal de [2] s'exprime ainsi : soit Ω un ouvert de $\mathbb{R}^N$. L'ensemble des valeurs prises par le jacobien généralisé sur $W^{1,k-1}(\Omega, S^{k-1})$ est exactement l'ensemble des frontières des courants rectifiables dans Ω qui ont une masse finie et qui sont de dimension $N - k + 1$.

0.6.2 Les contributions de cette thèse

0.6.2.1 Généralisation aux espaces de Sobolev $W^{s,p}, s \geq 1$

L'objectif du Chapitre 5 est de déterminer l'image par le jacobien généralisé de l'espace $W^{s,p}(S^N, S^1)$. Pour la clarté de l'exposé, je me restreins ici au cas $N = 2$, de sorte que les singularités topologiques sont des points. Lorsque $sp < 1$, Escobedo [31] a montré que $C^\infty(S^2, S^1)$ est dense dans $W^{s,p}(S^2, S^1)$. Lorsque $sp \geq 2$, le même résultat est vrai (voir [15] ou [20]). Il s'ensuit que dans ces deux cas, il n'y a pas de bonne notion d'ensemble singulier. Je me suis donc restreint au cas $1 \leq sp < 2$. Lorsque $s \geq 1$, on a l'inclusion $W^{s,p}(S^2, S^1) \subset W^{1,1}(S^2, S^1)$

et donc le jacobien généralisé peut être défini comme en (13). En particulier, pour tout élément $u \in W^{s,p}(S^2, S^1)$, il existe deux suites de points $(P_i), (N_i)$ dans S^2 telles que $\sum_i d(P_i, N_i) < \infty$ et

$$T(u) = \pi \sum_i (\delta_{P_i} - \delta_{N_i}) \tag{16}$$

(l'égalité est comprise dans $(W^{1,\infty}(S^2, \mathbb{R}))^*$). Cependant, pour toute suite $(P_i), (N_i)$ de S^2 telle que $\sum_i d(P_i, N_i) < \infty$, il n'existe pas nécessairement de fonction $u \in W^{s,p}(S^2, S^1)$ telle que (16) ait lieu. En d'autres termes, le jacobien généralisé T envoie $W^{s,p}(S^2, S^1)$ sur un ensemble strictement inclus dans $(W^{1,\infty}(S^2, \mathbb{R}))^*$.

Pour déterminer cet ensemble, introduisons l'ensemble $\mathcal{T}$ des distributions de la forme

$$\pi \sum_{\text{finie}} (\delta_{B_i} - \delta_{C_i})$$

où $(B_i), (C_i)$ sont deux familles finies de points de la sphère S^2. Par ailleurs, notons X l'espace vectoriel normé

$$X := (W^{2-s,p'}(S^2, \mathbb{R}))^* \cap (W^{1,(sp)'}(S^2, \mathbb{R}))^*,$$

avec $p' := p/(p-1)$ et $(sp)' := sp/(sp-1)$. Lorsque $s = 1$, X est simplement $(W^{1,p'}(S^2, \mathbb{R}))^*$. Alors on a

Théorème 0.6
Soit $s \geq 1, 1 < p < \infty, 1 < sp < 2$.
a) Pour tout $u \in W^{s,p}(S^2, S^1), T(u)$ appartient à l'adhérence de $\mathcal{T}$ dans X.
b) Réciproquement, si T appartient à l'adhérence de $\mathcal{T}$ dans X, alors il existe une fonction $u \in W^{s,p}(S^2, S^1)$ telle que $T(u) = T$.

Ce résultat est nouveau même pour $s = 1, p > 1$.

La partie a) se démontre de la manière habituelle (voir [9], [10], [22]) : on commence par montrer que l'ensemble des fonctions de $W^{s,p}(S^2, S^1)$ qui sont lisses sauf sur un ensemble fini de points est dense dans $W^{s,p}(S^2, S^1)$ (ce résultat de densité s'appuie sur une idée de [41], voir aussi [9]). Puis on prouve que T est une application continue de $W^{s,p}(S^2, S^1)$ à valeurs dans X. (Ce résultat de continuité repose sur les propriétés de multiplication dans les espaces de Sobolev fractionnaires, voir [78]).

La partie b) suit une ligne moins traditionnelle (voir néanmoins [51], [2]). Le problème se ramène à montrer que pour tout $T \in \mathcal{T}$, on peut trouver une fonction $u \in W^{s,p}(S^2, S^1)$ telle que $T(u) = T$ avec une estimation de $||u||_{W^{s,p}}$ en fonction de $||T||_{(W^{2-s,p'})^*}$ et $||T||_{(W^{1,(sp)'})^*}$. Habituellement, pour obtenir des résultats de surjectivité sur le jacobien, on utilise la méthode des dipoles introduite dans [19] (et réexploitées dans [9], [2]). Cette méthode ne peut pas s'appliquer dans le contexte des espaces $W^{s,p}$ si $sp \neq 1$ car elle ne permet pas d'obtenir les estimations requises sur $||u||_{W^{s,p}}$. A la place, on exploite le fait que la variété cible est S^1 et qu'on peut chercher u sous la forme $u = e^{i\phi}$ où $\phi : S^2 \to S^1$ est une fonction à variation bornée. Un autre outil est constitué par les estimations $W^{s,p}$ des solutions de l'équation $\Delta v = T$ (où T est donné) sur

S^2. La difficulté est alors de trouver $u = e^{i\phi}$ solution de l'équation $D(u) = \nabla v$ où $D(u)$ est donné par (14). On conclut alors en notant que $T(u) = \operatorname{div} D(u) = \operatorname{div} \nabla v = T$.

Une conséquence facile du Théorème 0.6 est donnée par

Théorème 0.7
On a l'équivalence

$$u \in \overline{C^\infty(S^2, S^1)}^{W^{s,p}(S^2,S^1)} \Longleftrightarrow T(u) = 0.$$

Le théorème 0.7 est un outil commode pour montrer qu'un élément u dans l'espace $W^{s,p}(S^2, S^1)$ ne peut être approché par des fonctions lisses : il suffit de calculer son jacobien.

0.6.2.2 Perspectives

Le théorème 0.6 a un énoncé quelque peu abstrait. Il serait intéressant d'avoir une condition plus concrète sur les points P_i, N_i dans S^2 pour qu'on puisse donner un sens à $\sum_i (\delta_{P_i} - \delta_{N_i})$ dans l' espace $(W^{2-s,p'})^* \cap (W^{1,(sp)'})^*$. Et en premier lieu, la question suivante demeure pour l'instant ouverte : si $(P_i), (N_i)$ sont deux suites finies de points dans $\mathbb{R}^2$, peut-on donner la valeur de

$$|| \sum_i (\delta_{P_i} - \delta_{N_i}) ||_{(W^{1,p'})^*}$$

en fonction de l'emplacement des points P_i et N_i ? (Cette question a trouvé sa réponse dans le concept de connexion minimale lorsque $p = 1$; ce concept peut-il être généralisé au cas $p > 1$?)

Dans les résultats du Chapitre 5, rien n'est dit sur les espaces de Sobolev fractionnaires $W^{s,p}(S^2, S^1)$ pour $0 < s < 1$. Or, on a mentionné que le jacobien généralisé avait été étendu jusqu'aux espaces $W^{1/p,p}(S^2, S^1)$. La détermination de l'image du jacobien sur de tels espaces reste ouverte. Cependant, Ponce ([75]) a annoncé avoir prouvé le Théorème 0.7 pour les valeurs $0 < s < 1$.

Enfin, qu'en est-il de l'image du jacobien généralisé lorsque la variété cible S^1 est remplacée par $S^k, k > 1$? La partie a) du théorème 0.6 semble pouvoir se généraliser sans difficulté à ce cas. En revanche, il n'en va pas de même de la partie b). En particulier, l'équation $D(u) = X$ (lorsque X est donné et vérifie $\operatorname{div} X = 0$) n'admet pas toujours de solutions (voir [19]).

Chapter 1

On the Lower Bounded Slope Condition

This chapter is based on the paper *On the lower bounded slope condition* accepted at the *Journal of Convex Analysis.*

1.1 Introduction

Hilbert-Haar theory is one of the classical approaches to regularity in the multiple integral calculus of variations. The classical version of the Hilbert-Haar theorem can be stated as follows. Let $n \geq 2, F : \mathbb{R}^n \to \mathbb{R}$ be a convex function and Ω a bounded open set in $\mathbb{R}^n$. Let $\phi : \Gamma = \partial\Omega \to \mathbb{R}$ be a function which satisfies a Bounded Slope Condition (BSC) of rank Q. The BSC of rank Q is the requirement that given any point γ on the boundary, there exist two affine functions

$$y \mapsto \langle \zeta_\gamma^-, y - \gamma \rangle + \phi(\gamma),\ y \mapsto \langle \zeta_\gamma^+, y - \gamma \rangle + \phi(\gamma)$$

agreeing with ϕ at γ whose slopes satisfy $|\zeta_\gamma^-|, |\zeta_\gamma^+| \leq Q$ and such that

$$\langle \zeta_\gamma^-, \gamma' - \gamma \rangle + \phi(\gamma) \ \leq\ \phi(\gamma') \ \leq\ \langle \zeta_\gamma^+, \gamma' - \gamma \rangle + \phi(\gamma) \quad \forall \gamma' \in \Gamma.$$

Then the functional $I : u \mapsto \int_\Omega F(\nabla u)$ has a minimum over all the Lipschitz functions which assume the boundary values ϕ on Γ.

When Hilbert or Haar gave their versions of this theorem (with $F(p) = |p|^2$, see [49] ; with $n = 2$, see [38]), they used a *three point condition* which is equivalent to the BSC (when $n = 2$). Hartman and Nirenberg [46] formulated the BSC (after Rado had done it for $n = 2$, see [76]) and Stampacchia [84] coined the term BSC and gave the first proof of the Hilbert-Haar theorem in dimensions greater than 2. The BSC has also been used in the context of elliptic pde's (see [44] , [87] and [35]).

Miranda published in [69] (see also [37] and [71]) the proof of the Hilbert-Haar theorem as stated above. One drawback of this theorem is that the BSC hypothesis is quite restrictive. First, if ϕ is not the restriction of a linear function, it implies that Ω is convex. Indeed, the BSC hypothesis implies the existence of a supporting hyperplane at any point γ of Γ, namely :

$$\{\gamma' \in \mathbb{R}^n : \langle \zeta_\gamma^- - \zeta_\gamma^+, \gamma' - \gamma \rangle = 0\}.$$

Secondly, the BSC hypothesis forces ϕ to be a Lipschitz function and to be affine on any segment in Γ. Additionally, Hartman [43] has shown that if Γ is smooth, then any ϕ satisfying the BSC must be smooth. (A precise statement appears below.)

All this has led Clarke [28] to introduce a new property so as to generalize Hilbert-Haar theory to a wider class of boundary functions, namely those functions which satisfy the so-called Lower Bounded Slope Condition (LBSC). The aim of this article is to understand how wide this class is and to characterize it.

Definition 1.1 *The function $\phi : \Gamma \to \mathbb{R}$ is said to satisfy the LBSC of rank Q if given any $x \in \Gamma$, there exists an affine function*

$$y \mapsto \langle \zeta_x, y - x \rangle + \phi(x)$$

with $|\zeta_x| \leq Q$ such that

$$\langle \zeta_x, y - x \rangle + \phi(x) \leq \phi(y) \quad \forall y \in \Gamma.$$

The following proposition gives a first characterization of functions satisfying the LBSC.

Proposition 1.1 *The function $\phi : \Gamma \to \mathbb{R}$ satisfies the LBSC if and only if it is the restriction to Γ of a (finite) convex function.*

In contrast, it is known that functions satisfying the BSC are precisely those which coincide on Γ with a convex function and also with a concave function (see [43]). The proof of Proposition 1.1 is given in section 3.

Actually, the proof will show that ϕ satisfies the LBSC of rank Q if and only if it is the restriction to Γ of a convex function which is globally Lipschitz of rank Q. As an example, one can show that the functions satisfying the LBSC on a square are the Lipschitz functions which are convex on each side of the square (see the appendix of Chapter 1 for a proof).

We can also define the Upper Bounded Slope Condition (UBSC) which is satisfied by $\phi : \Gamma \to \mathbb{R}$ exactly when $-\phi$ satisfies the LBSC. Note that ϕ satisfies the BSC if and only if ϕ satisfies the LBSC and the UBSC.

Though the BSC forces boundary functions to be affine on flat parts of the boundary, it becomes more interesting when Ω is sufficiently curved.

Definition 1.2 *A convex set Ω is said to be uniformly convex if, for some $\epsilon > 0$, for every point γ on the boundary, there exists a unit vector $b_\gamma \in \mathbb{R}^n$ such that*

$$\langle b_\gamma, \gamma' - \gamma \rangle \geq \epsilon |\gamma' - \gamma|^2, \quad \forall \gamma' \in \Gamma.$$

Miranda's Theorem [69] states that when Ω is uniformly convex, then any ϕ of class C^2 (and actually $C^{1,1}$ is enough) satisfies the BSC. We can prove an analogue of this for functions satisfying the LBSC. The LBSC requires only the minoration inequality of the two inequalities defining the BSC. In that sense, the LBSC is a one-sided BSC. It turns out that the one-sided $C^{1,1}$ regularity (that is regularity required only "from below") is exactly semiconvexity (or equivalently, up to sign, semiconcavity, a familiar and useful property in pde's, see [23]).

Definition 1.3 *Let S be a subset of $\mathbb{R}^m$. The function $u : S \subset \mathbb{R}^m \to \mathbb{R}$ is said to be semiconvex if there exists a lower semicontinuous function $\omega : \mathbb{R}^+ \to \mathbb{R}^-$ which is nonincreasing, such that $\lim_{\rho \to 0^+} \omega(\rho) = 0$ and*

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda y)) \geq \lambda(1-\lambda)|x-y|\omega(|x-y|)$$

for any $x, y \in S$ such that $[x, y] \subseteq S$ and for any $\lambda \in [0, 1]$. We call such an ω a modulus of semiconvexity for u on S.

We say that a function is locally semiconvex if it is semiconvex on every compact subset of its domain of definition. A function is said to be [locally] semiconcave if its negative is [locally] semiconvex.

Finally, if ω is of the form $-C|\cdot|$ where $C \geq 0$, we say that u is linearly semiconvex.

This definition implies that convex functions are semiconvex functions with a vanishing modulus of semiconvexity $\omega = 0$. Actually, u is linearly semiconvex on an open convex set S with modulus of convexity $-C|\cdot|$ if and only if $u + C/2|\cdot|^2$ is convex on S (see [23], Proposition 1.1.3). Semiconvex functions share with convex functions the property of being locally Lipschitz.

Then we have the following:

Proposition 1.2 *When Ω is uniformly convex, ϕ satisfies the LBSC if and only if it is the restriction to Γ of a function which is locally linearly semiconvex on $\mathbb{R}^n$.*

See section 3 for a proof of Proposition 1.2.

In 1966, Hartman [43] found a converse to Miranda's earlier result.

Theorem 1.1 *Let Ω be a bounded open convex set and ϕ a function on $\Gamma = \partial\Omega$ satisfying a BSC. If Γ is C^1, then ϕ is C^1. If Γ is $C^{1,\lambda}$ for some $\lambda \in]0, 1]$, then ϕ is $C^{1,\lambda}$.*

A function on an open set $A \subset \mathbb{R}^n$ is said to be of class $C^{1,\lambda}$ if it has continuous first order partial derivatives which are uniformly Hölder [or Lipschitz] continuous of order $\lambda, 0 < \lambda < 1$ [or $\lambda = 1$] on closed balls in A. A hypersurface $\Gamma \subset \mathbb{R}^n$ is said to be of class $C^{1,\lambda}$ if for any $x \in \Gamma$, there exists a parametrization $\rho : V \to \Gamma \cap U \ni x$ (that is V is an open set in $\mathbb{R}^{n-1}, U$ is an open set in $\mathbb{R}^n$ containing x and ρ is an immersion and a homeomorphism onto its image) which is of class $C^{1,\lambda}$. Finally, $\phi : \Gamma \to \mathbb{R}$ is said to be $C^{1,\lambda}$ if for any such parametrization $\rho : V \to U, \phi \circ \rho$ is of class $C^{1,\lambda}$. We will give in Section 3 a (new) short proof of Theorem 1.1, based on the natural link between the LBSC and semiconvexity. But it is a natural question to ask whether such a result still holds if one replaces (for ϕ) BSC by LBSC and $C^{1,\lambda}$ by semiconvexity. The map $\phi : \Gamma \to \mathbb{R}$ is said to be [linearly] *semiconvex* if $\phi \circ \rho$ is locally [linearly] semiconvex on the open set $V \subset \mathbb{R}^{n-1}$ (in the sense of Definition 1.3) for any parametrization $\rho : V \to U \cap \Gamma$. We will prove in Section 3 the following :

Theorem 1.2 *Let Ω be a bounded open convex set, $\Gamma := \partial\Omega$ being $C^{1,1}$ and ϕ a function on Γ. If ϕ satisfies the LBSC, then ϕ is linearly semiconvex. If moreover Ω is uniformly convex, then the converse is true; that is, if ϕ is linearly semiconvex, then ϕ satisfies the LBSC.*

The first part of the theorem is the counterpart for LBSC functions of Theorem 1.1. The last assertion does not coincide with Proposition 1.2 as it is of a local nature. It is a general principle in convexity theory that local properties are simultaneously global (see for instance Claim 1.3 in the proof of Theorem 1.2). This is not the case for semiconvexity on hypersurfaces in $\mathbb{R}^n$.

Even if the two articles [43], [45] deal with the BSC, most of the proofs stated there are valid for the LBSC. We enumerate some of these results (for the LBSC) in section 2 as well as a new result concerning continuity of minimizers in the multiple integral calculus of variations, see Theorem 1.5 below.

In Section 3, we prove Proposition 1.1 and 1.2 as well as Theorem 1.1 and Theorem 1.2, and underline the local nature of the LBSC. In Section 4, we will provide intrinsic characterizations of the LBSC in terms of subgradients of ϕ.

1.2 Some further results

We state now some further results about the LBSC, which can be deduced from the proofs appearing in [43] and [45]. In section 3, Theorem 1.3 will be used to show that a particular example of a function ϕ does not satisfy the LBSC.

The proof by Hartman of Theorem 1.1 has a geometrical flavour, and one of the main results in [43] states an equivalence between the BSC and the $n+1$ points condition. (Actually, the case $n = 2$ had been known for a long time, see [36] for a proof in this case.) We say that ϕ satisfies an $n+1$ points condition [with constant K] if for every set of $n+1$ points $x_0, ...x_n$ of Γ, there is a hyperplane $z = \langle a, x\rangle + c = \sum_{h=1}^{n} a^h x^h + c$ in $\mathbb{R}^{n+1}$ which passes through the points $(x, z) = (x_j, \phi(x_j))$ for $j = 0, ..., n$ and satisfies

$$|a| := (\sum_{k=1}^{n} |a^k|^2)^{1/2} \leq K.$$

It is easy to see that the same proof as in [45], Theorem 3.1 yields a similar result for functions satisfying the LBSC:

Proposition 1.3 *The function ϕ satisfies the LBSC if and only if there exists $S \in \mathbb{R}$ such that for every set of $n+1$ linearly independent points $x_0, ...x_n$ in Γ, the couple $(a, c) \in \mathbb{R}^n \times \mathbb{R}$ defined by*

$$\phi(x_i) = \langle a, x_i\rangle + c \quad \forall i = 0, ..., n$$

satisfies $\langle a, x\rangle + c \geq S, \forall x \in \Omega$.

In other words, any hyperplane in $\mathbb{R}^{n+1}$ passing through the points $(x_i, \phi(x_i))$ lies above the *horizontal* hyperplane $z = S$ on Ω. The proof of this proposition is based on a quite effective characterization of functions satisfying the LBSC, which we now describe (see [43], Corollary 2.1 which holds for BSC functions but whose proof is also valid for LBSC functions).

Let $x_* \in \Omega$ be fixed and x_0, x_1 be distinct points in Γ. By a point x_{01} of Γ *between* x_0 and x_1 is meant a point of the form $x_{01} = x_* + \lambda(x_0 - x_*) + \mu(x_1 - x_*)$ with $\lambda, \mu > 0$. In the 2 dimensional plane π defined by the three points

x_*, x_0, x_1, introduce rectangular coordinates (ξ, η) with x_* as origin such that if $(\xi_0, \eta_0), (\xi_1, \eta_1)$ denote the coordinates of x_0, x_1 respectively, then

$$\begin{vmatrix} \xi_0 & \eta_0 \\ \xi_1 & \eta_1 \end{vmatrix} > 0$$

(in other words, the basis defining the coordinates (ξ, η) has the same orientation in π as the basis $(x_0 - x_*, x_1 - x_*)$). Then we have

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set, $\phi : \Gamma = \partial\Omega \to \mathbb{R}$, and $x_* \in \Omega$. Then ϕ satisfies a LBSC if and only if there exists a number S such that for $z_* \leq S$, the inequality*

$$\begin{vmatrix} \xi_0 & \eta_0 & \phi(x_0) - z^* \\ \xi_{01} & \eta_{01} & \phi(x_{01}) - z^* \\ \xi_1 & \eta_1 & \phi(x_1) - z^* \end{vmatrix} \geq 0 \tag{1.1}$$

holds for all points $x_0, x_1 \in \Gamma$ and points x_{01} between them, $(\xi_0, \eta_0), (\xi_1, \eta_1)$ and (ξ_{01}, η_{01}) being the coordinates of x_0, x_1, x_{01} respectively.

Two years later, Hartman [45] made another significant contribution to the understanding of the BSC. Let Ω be a bounded open convex set. Let $\Lambda(\Gamma)$ be the set of all those $\phi : \Gamma \to \mathbb{R}$ such that ϕ is continuous on Γ and on every line segment $l \subset \Gamma, \phi_{|l}$ is the restriction of an affine function. Then

Theorem 1.4 *The set of all those ϕ satisfying the BSC is dense in $\Lambda(\Gamma)$ for the uniform norm.*

This result enabled Hartman to generalize Miranda's Theorem [69] concerning generalized solutions of the Dirichlet boundary value problem for the minimal surface equation and a continuous boundary function on a uniformly convex set. Actually, as seen by Hartman, this theorem and its proof still hold when Ω is an arbitrary bounded open convex set and ϕ is in the closure in $C^0(\Gamma)$ of the set of those functions satisfying the BSC. Indeed, the proof of Miranda's theorem is based on an approximation procedure and the Hilbert-Haar theorem. It is a striking feature of the Hilbert-Haar theory, that applying it to a sequence of problems can give useful information for a limit problem, associated with a boundary function which does not satisfy the BSC or the LBSC (see [69], [64] and [57]). We give here a regularity result of this kind in the multiple integral calculus of variations. To our knowledge, this result is new.

Theorem 1.5 *Let Ω be a bounded open convex set in $\mathbb{R}^n, \phi \in \Lambda(\Gamma)$ and $I(u) := \int_\Omega F(\nabla u)$. Here, F is a strictly convex function on $\mathbb{R}^n$. We consider the problem of minimizing I over the functions $u \in W^{1,1}(\Omega)$ that assume boundary values ϕ. If u is a solution, then it is continuous on $\bar{\Omega}$.*

Proof : Let ϕ_i be a sequence of functions satisfying the BSC and uniformly converging to ϕ on Γ (Theorem 1.4 provides the existence of this sequence). The Hilbert-Haar theorem yields the existence of a Lipschitz function u_i which minimizes I relative to all Lipschitz functions having value ϕ_i on Γ. Mariconda and Treu [61] have shown that no Lavrentiev phenomenon can occur; that is, u_i minimizes I over all $v \in W_0^{1,1}(\Omega) + \phi$. To see this, apply the main theorem in

[26] which yields the existence of a bounded function k which is a measurable selection of the convex subgradient of F along $\nabla u_i(x) : k(x) \in \partial F(\nabla u_i(x))$ a.e. on Ω, such that

$$\int_\Omega \langle k(x), \nabla\eta(x)\rangle \, dx = 0 \quad \forall \eta \in C_c^\infty(\Omega)$$

and then (as k is bounded) this remains true for any $\eta \in W_0^{1,1}(\Omega)$. So

$$I(u_i + \eta) \geq I(u_i) + \int_\Omega \langle k(x), \nabla\eta(x)\rangle \, dx = I(u_i)$$

in view of the definition of the convex subgradient.

Now, u and u_i being minimisers of I, we can apply the comparison principle stated in [62] to deduce

$$|u(x) - u_i(x)| \leq ||\phi - \phi_i||_{L^\infty(\Gamma)} \quad \forall x \in \Omega.$$

So, the sequence u_i is a Cauchy sequence in $C^0(\bar{\Omega})$ which converges to a continuous representative of u. This completes the proof.

□

The proof of Theorem 1.4 by Hartman (see Proposition 3.5 in [45]) shows in particular that:

Theorem 1.6 *The set of those functions $\phi : \Gamma \to \mathbb{R}$ satisfying the LBSC is dense (for the uniform norm) in the subset of those continuous functions which are convex on any line segment $l \subset \Gamma$.*

1.3 The Lower Bounded Slope Condition and Semiconvexity

First, we show the characterization of the LBSC given in the Introduction which makes a link between the LBSC and convexity.

Proof of Proposition 1.1: If ϕ is the restriction to Γ of a convex function $\tilde{\phi}$ defined on $\mathbb{R}^n$, then for every $x \in \Gamma$, there exists ζ in the convex subgradient of $\tilde{\phi}$ at $x, \zeta \in \partial\phi(x)$, which means

$$\tilde{\phi}(y) \geq \tilde{\phi}(x) + \langle \zeta, y - x\rangle \quad \forall y \in \mathbb{R}^n.$$

Since a convex function is locally Lipschitz, there exists some $Q \geq 0$ such that $\tilde{\phi}$ is Q Lipschitz on a neighborhood of $\bar{\Omega}$, which implies $\partial\tilde{\phi}(x) \subset \bar{B}(0, Q)$. Hence, there exists $Q \geq 0$ such that for every $x \in \Gamma$, there exists $\zeta \in \mathbb{R}^n$ such that

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x\rangle \quad \forall y \in \Gamma,$$

which is the LBSC of rank Q. Conversely, if ϕ satisfies the LBSC of rank Q, then let us define

$$\Phi(y) := \sup_{x\in\Gamma}(\phi(x) + \langle \zeta_x, y - x\rangle),$$

where $\zeta_x \in \mathbb{R}^n$ is such that

$$\phi(y) \geq \phi(x) + \langle \zeta_x, y - x\rangle, \forall y \in \Gamma$$

and $|\zeta_x| \leq Q$. Then, the supremum is finite and no greater than $\phi(y)$. Moreover, for any $y \in \Gamma, \phi(y) = \phi(y) + \langle \zeta_y, y - y \rangle$, so that $\Phi(y) \geq \phi(y)$. So ϕ is the restriction to Γ of Φ, which is a convex function as the supremum of affine functions.

□

Proposition 1.2 improves this result when Ω is uniformly convex.

Proof of Proposition 1.2 : The *only if* part is obvious in view of the fact that convex functions are semiconvex, and in view of Proposition 1.1 . It does not require the uniform convexity of Ω. Conversely, if ϕ is the restriction to Γ of a locally linearly semiconvex function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}$, then there exists $C, Q \geq 0$ such that for all $x \in \Gamma$, there exists $\zeta_x \in \bar{B}(0, Q)$ satisfying

$$\phi(y) \geq \phi(x) + \langle \zeta_x, y - x \rangle - C|y - x|^2 \quad \forall y \in \Gamma.$$

(see [23], Propositions 3.3.1 and 3.3.4). Here, ζ_x is a Frechet subgradient to $\tilde{\phi}$ at $x, -C|\cdot|$ is a modulus of semiconvexity on some neighborhood of $\bar{\Omega}$ and Q is a Lipschitz constant for $\tilde{\phi}$ on this neighborhood. Furthermore, by uniform convexity, there exists $\epsilon > 0$ and a unit vector $b_x \in \mathbb{R}^n$ such that

$$\langle b_x, y - x \rangle \geq \epsilon |y - x|^2 \quad \forall y \in \Gamma.$$

When put together, these inequalities imply

$$\phi(y) \geq \phi(x) + \langle \zeta_x - \frac{C}{\epsilon} b_x, y - x \rangle \ \ \forall y \in \Gamma.$$

Therefore ϕ satisfies the LBSC of rank $Q + \frac{C}{\epsilon}$. This completes the proof of Proposition 1.2.

□

As said before, semiconvexity is a useful tool to deal with the LBSC. The two following propositions will be crucial in the sequel (the first one is exactly Theorem 3.3.7 in [23] whereas the second one corresponds to Proposition 2.1.12 and its proof there).

Proposition 1.4 *If $u : V \to \mathbb{R}$, with V open, is both semiconvex and semiconcave in V, then $u \in C^1(V)$. Moreover, if the moduli of semiconvexity and semiconcavity of u both have the form $\omega(r) = Cr^\alpha$, for the same $\alpha \in]0, 1]$, then $u \in C^{1,\alpha}(V)$.*

The same is true when semiconvex [semiconcave] is replaced by locally semiconvex [locally semiconcave] and the moduli of semiconvexity [semiconcavity] depend on the compact subset $S \subset V$. Recall that $C^{1,\alpha}$ in this article means the derivative is Hölderian on any closed ball in V (and not necessarily globally on V).

Proposition 1.5 *Let $u : A \to \mathbb{R}$ be a locally semiconvex function on an open set A and $\rho : V \to A$ a function of class C^1 on an open set V of $\mathbb{R}^{n-1}$. Then $u \circ \rho$ is locally semiconvex on V. More precisely, if S is a compact subset of V, such that $\mathrm{co}[\rho(S)] \subset A$, then $L_1\omega_2(\cdot) + \omega_1(L_2 \cdot)$ is a modulus of semiconvexity of $u \circ \rho$ on S where ω_2 [resp. ω_1] is the modulus of continuity of $D\rho$ on S [resp. the modulus of semiconvexity of u on $\mathrm{co}[\rho(S)]$ and L_1 [resp. L_2] a Lipschitz constant for u on $\rho(S)$ [resp. of ρ on S].*

Theorem 1.1 is an easy consequence of the properties satisfied by semiconvex functions.

Proof of Theorem 1.1: Let $\rho : V \to U$ be a parametrization of class $C^{1,\lambda}$. We must show that $\phi \circ \rho$ is of class $C^{1,\lambda}$. Since ϕ satisfies the *LBSC*, it is the restriction of a convex function $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}$. Let S be any compact subset of V and let $\hat{L} \in \mathbb{R}$ be a Lipschitz constant for $\hat{\phi}$ on a neighborhood of $\mathrm{co}[\rho(S)]$. Then, thanks to Proposition 1.5, $\phi \circ \rho = \hat{\phi} \circ \rho$ is semiconvex on S, a modulus of semiconvexity being $\hat{L}\omega_S$ where ω_S is the modulus of continuity of $D\rho$ on S. Using now the fact that ϕ is UBSC, we can find in a similar way that there exists $\check{L} \in \mathbb{R}$ such that $\phi \circ \rho$ is semiconcave on S with modulus of semiconcavity $\check{L}\omega_S$.

Since ρ is $C^{1,\lambda}$, the modulus of continuity of $D\rho$ on S is of the form $\omega_S(r) = C_S|r|^\lambda$ with some $C_S \geq 0$. Proposition 1.4 then implies that $\phi \circ \rho$ is of class $C^{1,\lambda}$ on $\mathrm{int}\, S$. This shows that $\phi \circ \rho$ is $C^{1,\lambda}$ on V and completes the proof. $\square$

Proof of Theorem 1.2: If Γ is $C^{1,1}$ and ϕ satisfies the LBSC, then ϕ is the restriction to Γ of a convex function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}$, so that $\phi \circ \rho = \tilde{\phi} \circ \rho$ is locally linearly semiconvex for any parametrisation $\rho : V \subset \mathbb{R}^n \to U \cap \Gamma$ (thanks to Proposition 1.5). This means that ϕ is linearly semiconvex.

The converse is not so straightforward, as the semiconvexity of ϕ is a local property and the LBSC appears (as far as its definition is concerned) as a global property (involving all of Γ).

Definition 1.4 *If U is an open set in $\mathbb{R}^n$, we say that $\phi_{|U}$ satisfies the LBSC if there exists $Q \geq 0$ such that for any $x \in U \cap \Gamma$, there exists $\zeta_x \in \bar{B}(0, Q)$ such that $\phi(y) \geq \phi(x) + \langle \zeta_x, y - x \rangle, \forall y \in \Gamma \cap U$.*

Fix some x_* in Ω. For each $x \in \Gamma$, there exists a parametrization $\rho : V \subset \mathbb{R}^{n-1} \to U \cap \Gamma \ni x$ which is $C^{1,1}$. Moreover,

Claim 1.1 *For any $x \in \rho(V)$, there exists $U_1 \subset U, V_1 \subset \bar{V}_1 \subset V$, with V_1 an open convex set and U_1 an open set, $\psi : U_1 \to V_1$ of class $C^{1,1}$ such that*

$$\rho \circ \psi(x') = x' \ \forall x' \in \Gamma \cap U_1, \quad \psi \circ \rho(v') = v', \ \forall v' \in V_1. \tag{1.2}$$

Claim 1.1 is an easy consequence of the Inverse Function Theorem applied to the function $\tilde{\rho} : (v, t) \in V \times \mathbb{R} \mapsto \rho(v) + tn$ where n is any vector in $\mathbb{R}^n$ not belonging to $D\rho(\rho^{-1}(x))\mathbb{R}^{n-1}$. There exist $\epsilon > 0$, an open convex set V_1 in $\mathbb{R}^{n-1}$ containing $\rho^{-1}(x)$, and an open set U_0 in $\mathbb{R}^n$ containing x such that $\tilde{\rho}$ is a $C^{1,1}$ diffeomorphism from $V_1 \times]-\epsilon, \epsilon[$ onto U_0. Define an open set $U_1 \subset U_0$ such that $U_1 \cap \Gamma = \rho(V_1)$, and $\psi := \Pi \circ \tilde{\rho}^{-1} : U_1 \to V_1$, where

$$\Pi : (v_1, ...v_{n-1}, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \to (v_1, ..., v_{n-1}) \in \mathbb{R}^{n-1}.$$

Then (1.2) holds.

For any $x \in \rho(V)$, and U_1, V_1, ψ as in Claim 1.1, $\phi \circ \rho$ is linearly semiconvex and Lipschitz on V_1. Hence, $\phi \circ \rho \circ \psi$ is locally linearly semiconvex on U_1. As $\phi \circ \rho \circ \psi = \phi$ on $\Gamma \cap U_1$, we see that ϕ is the restriction to $\Gamma \cap U_1$ of a locally linearly semiconvex function defined on U_1. Therefore, using the fact that Ω is uniformly convex (exactly as in Proposition 1.2), $\phi_{|U_x}$ satisfies the LBSC for any open set U_x satisfying $x \in U_x \subset \bar{U}_x \subset U_1$.

When x runs through Γ, the corresponding sets U_x constitute a covering of the compact set Γ. We extract from this covering a finite one which we will denote for ease of notation $U_1, .., U_m$, and correspondingly $\Xi_1, ..., \Xi_m$, will denote the following open subsets of $\mathbb{R}^n$:

$$\Xi_i := \{x_* + t(x - x_*) : t > 0, x \in U_i\}.$$

Finally, Γ_i will denote $U_i \cap \Gamma$. Theorem 1.2 will be then a direct consequence of the following lemma, which is of independent interest.

Lemma 1.1 *If for any $i = 1, ..., m, \phi_{|U_i}$ satisfies the LBSC of rank Q_i, then ϕ satisfies the LBSC.*

We now prove Lemma 1.1. For each $z_* \in \mathbb{R}^-$, we consider the function (as in [43])

$$\tau_{z_*} : x \in \mathbb{R}^n \to z = z_* + t[\phi(x_0) - z_*] \tag{1.3}$$

where (t, x_0) is defined as :

if $x \neq x_*, x_0$ is the unique point of Γ of the form $x_* + s(x - x_*)$, with $s > 0$ and t is defined by $x = x_* + t(x_0 - x_*)$;

if $x = x_*$, we set $t = 0$ (and x_0 any point in Γ). In the notation of [28], $x_0 = \pi_\Gamma(x_*|x)$ and $t = |x - x_*|/d_\Gamma(x_*|x)$.

Claim 1.2 *For any $i = 1, ..., m$, if $\phi_{|U_i}$ satisfies the LBSC of rank Q_i, then there exists $N_i \in \mathbb{R}$ such that for any $z_* \leq N_i, \tau_{z_*|\Xi_i}$ is convex.*

This claim is a local one-sided version of Theorem 2.1 in [43].

Proof of Claim 1.2: For every $x \in \Gamma_i$, there exists $\zeta_x \in \bar{B}(0, Q_i)$ such that $\phi(y) \geq \phi(x) + < \zeta_x, y - x >=: v_x(y)$, $\forall y \in \Gamma_i$. There exists a number $N_i \leq -||\phi||_\infty$ (depending only on $||\phi||_{L^\infty}, Q_i, \operatorname{diam}\Omega$, where the latter denotes $\sup_{x,y\in\Omega} |x-y|$) such that $v_x(x_*) \geq N_i$ for any $x \in \Gamma_i$. Let $z_* \leq N_i$. Let $a_x \neq 0$ be in the convex cone to Ω at x and $\mu_x \geq 0$ such that

$$v_x(x_*) + \langle \mu_x a_x, x_* - x \rangle = z_*$$

(μ_x certainly exists since $< a_x, x_* - x >< 0$). Then we claim that

$$\tau_{z_*}(y) = \sup_{x\in\Gamma_i} (v_x(y) + \langle \mu_x a_x, y - x \rangle) \quad \forall y \in \Xi_i.$$

Indeed, let $x_0 \in \Gamma_i, t \geq 0$ and $y = x_* + t(x_0 - x_*)$. Then

$$\begin{aligned} \tau_{z_*}(y) = z_* + t(\phi(x_0) - z_*) &= v_{x_0}(y) + \langle \mu_{x_0} a_{x_0}, y - x_0 \rangle \\ &\leq \sup_{x\in\Gamma_i} (v_x(y) + \langle \mu_x a_x, y - x \rangle). \end{aligned}$$

And for any $x \in \Gamma_i$,

$$\begin{aligned} v_x(y) + \langle \mu_x a_x, y - x \rangle &= (1 - t)(v_x(x_*) + \langle \mu_x a_x, x_* - x \rangle) \\ &\quad + t(\underbrace{v_x(x_0)}_{\leq\phi(x_0)} + \underbrace{\langle \mu_x a_x, x_0 - x \rangle}_{\leq 0}) \\ &\leq (1 - t)z_* + t\phi(x_0). \end{aligned}$$

Hence, τ_{z^*} is a convex function on Ξ_i as the supremum of affine functions (though Ξ_i might not be convex). The conclusion of Claim 1.2 follows from that.
We can now finish the proof of Lemma 1.1. Indeed, setting $N := \min_{1\leq i\leq m} N_i$, we have for any $z_* \leq N$, that $\tau_{z_*|\Xi_i}$ is convex.

The following remark is useful.

Claim 1.3 *Let I be a nontrivial interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a locally convex function, in the following sense: for any $x \in I$, there exists $\epsilon > 0$ such that f restricted to $(x-\epsilon, x+\epsilon) \cap I$ is convex. Then f is convex on I.*

This is a well known fact which we admit. Now let x, x' be two points in $\mathbb{R}^n$. If $x_* \in (x, x')$, then $\tau_{z_*|(x_*,x)}$ is affine and so is $\tau_{z_*|(x_*,x')}$, with $\tau_{z_*}(x_*) \leq \tau_{z_*}(x), \tau_{z_*}(x')$. Then $\tau_{z_*|(x,x')}$ is convex. In the other case, $(x, x') \not\ni x_*$ and then $(x, x') = \bigcup_{i\in\{1,\ldots,m\}}((x, x')\cap\Xi_i))$, so that $\tau_{z_*|(x,x')}$ is locally convex, hence convex. As the restriction of τ_{z_*} to any straight line is convex, τ_{z_*} itself is convex. This shows that ϕ is the restriction to Γ of a convex function, and thus satisfies the LBSC.

□

Let us give an application of these results. The following example is used in [28] to show that even for the Dirichlet Lagrangian on the open disk, minimizers are not necessarily globally Lipschitz when the boundary function satisfies only the LBSC (but local Lipschitz continuity in the interior is obtained). We now show that the function involved satisfies the LBSC but not the BSC.

Example 1.1 *Let $\Omega \subset \mathbb{R}^2$ be the unit disc and*

$$\phi : (x, y) \in \Gamma \mapsto -\frac{\pi^2}{6} + \frac{\pi}{2}\theta - \frac{\theta^2}{4}.$$

where $\theta \in [0, 2\pi[$ is such that $(x, y) = (\cos\theta, \sin\theta)$. Then ϕ is a Lipschitz function satisfying the LBSC but not the BSC.

We will use $\rho : \theta \in \mathbb{R} \mapsto (\cos\theta, \sin\theta)$, which is a parametrization when restricted to any interval of length less than 2π. We have then $\phi\circ\rho : \theta \mapsto -\frac{\pi^2}{6} + \frac{\pi}{2}\theta - \frac{\theta^2}{4}$ when the right hand side is extended by 2π periodicity all over $\mathbb{R}$. The derivative of $\phi\circ\rho$ has a discontinuity at each point of the form $2k\pi, k \in \mathbb{Z}$. On $\Gamma\backslash\{(1, 0)\}$, ϕ is smooth so that ϕ restricted to $\Gamma\backslash[1/2, +\infty[\times\mathbb{R}$ satisfies the LBSC (see the proof of Proposition 1.2). To show that ϕ satisfies the LBSC, it is enough to check that there exists $\sigma \geq 0$ such that

$$\forall\theta_0 \in]-\pi, \pi[, \exists\zeta \in \bar{B}(0, \pi/2)$$

such that for any $\theta \in]-\pi, \pi[$

$$\phi\circ\rho(\theta) \geq \phi\circ\rho(\theta_0)+ < \zeta, \theta-\theta_0 > -\sigma|\theta-\theta_0|^2$$

which is equivalent to verifying four cases

$$\underbrace{\frac{\pi}{2}(\theta-\theta_0) - \frac{\theta^2}{4} + \frac{\theta_0^2}{4}}_{(\frac{\pi}{2}-\frac{\theta_0}{2})(\theta-\theta_0)-\frac{1}{4}(\theta-\theta_0)^2} \geq \langle\zeta, \theta-\theta_0\rangle - \sigma|\theta-\theta_0|^2 \quad \forall\theta, \theta_0 \in [0, \pi[,$$

$$\underbrace{\frac{\pi}{2}(-\theta+\theta_0)-\frac{\theta^2}{4}+\frac{\theta_0^2}{4}}_{(-\frac{\pi}{2}-\frac{\theta_0}{2})(\theta-\theta_0)-\frac{1}{4}(\theta-\theta_0)^2}\geq\langle\zeta,\theta-\theta_0\rangle-\sigma|\theta-\theta_0|^2\quad\forall\theta,\theta_0\in]-\pi,0],$$

$$\frac{\pi}{2}(\theta+\theta_0)-\frac{\theta^2}{4}+\frac{\theta_0^2}{4}\geq\langle\zeta,\theta-\theta_0\rangle-\sigma|\theta-\theta_0|^2\quad\forall\theta\in[0,\pi[,\theta_0\in]-\pi,0],$$

$$\frac{\pi}{2}(-\theta-\theta_0)-\frac{\theta^2}{4}+\frac{\theta_0^2}{4}\geq\langle\zeta,\theta-\theta_0\rangle-\sigma|\theta-\theta_0|^2\quad\forall\theta_0\in[0,\pi[,\theta\in]-\pi,0].$$

In the third case,

$$\begin{aligned}\frac{\pi}{2}(\theta+\theta_0)-\frac{\theta^2}{4}+\frac{\theta_0^2}{4} &= \pi\theta_0+(\theta-\theta_0)(\frac{\pi}{2}-\frac{\theta_0}{2})-\frac{1}{4}(\theta-\theta_0)^2\\ &\geq -\pi(\theta-\theta_0)+(\theta-\theta_0)(\frac{\pi}{2}-\frac{\theta_0}{2})-\frac{1}{4}(\theta-\theta_0)^2\\ &= (\theta-\theta_0)(-\frac{\pi}{2}-\frac{\theta_0}{2})-\frac{1}{4}(\theta-\theta_0)^2.\end{aligned}$$

In the last case, similarly,

$$\frac{\pi}{2}(-\theta-\theta_0)-\frac{\theta^2}{4}+\frac{\theta_0^2}{4}\geq(\frac{\pi}{2}-\frac{\theta_0}{2})(\theta-\theta_0)-\frac{(\theta-\theta_0)^2}{4}.$$

When $\theta_0\geq 0$, (that is, in the first and the last case), we can take $\zeta=\frac{\pi}{2}-\frac{\theta_0}{2}$ and when $\theta_0\leq 0$, (the second and third case), we can take $\zeta=-\frac{\pi}{2}-\frac{\theta_0}{2}$. When $\theta_0=0$, any of these two values of θ_0 will do.

Let us show now that $-\phi$ does not satisfy the LBSC, by contradicting Theorem 1.3. For any $\epsilon>0$, fix $x_0=(\cos\epsilon,-\sin\epsilon), x_{01}=(1,0)$ and $x_1=(\cos\epsilon,\sin\epsilon)$. Then with $x_*=(0,0)$,

$$\begin{aligned}\frac{\begin{vmatrix}\xi_0 & \eta_0 & -\phi(x_0)\\ \xi_{01} & \eta_{01} & -\phi(x_{01})\\ \xi_1 & \eta_1 & -\phi(x_1)\end{vmatrix}}{\begin{vmatrix}\xi_0 & \eta_0 & 1\\ \xi_{01} & \eta_{01} & 1\\ \xi_1 & \eta_1 & 1\end{vmatrix}} &= \frac{(\frac{\pi^2}{6}-\frac{\pi}{2}\epsilon+\frac{\epsilon^2}{4})\sin\epsilon-\frac{\pi^2}{6}\sin\epsilon\cos\epsilon}{\sin\epsilon(1-\cos\epsilon)}\\ &= \frac{\frac{\pi^2}{6}\sin\epsilon(1-\cos\epsilon)+(-\frac{\pi}{2}\epsilon+\frac{\epsilon^2}{4})\sin\epsilon}{\sin\epsilon(1-\cos\epsilon)}\\ &= \frac{\pi^2}{6}+\frac{-\frac{\pi\epsilon}{2}+\frac{\epsilon^2}{4}}{1-\cos\epsilon}.\end{aligned}$$

As $1-\cos\epsilon\sim\frac{\epsilon^2}{2},\epsilon\to 0$, the last term tends to $-\infty$ when $\epsilon\to 0$. This prevents Hartman's criterion (1.1) from holding. So $-\phi$ does not satisfy the LBSC, and ϕ fails to satisfy the BSC.

1.4 Subgradients

The aim of this section is to give intrinsic characterizations (i.e. without any parametrization) for a function $\phi:\Gamma\to\mathbb{R}$ to satisfy the LBSC. This characterization is in term of subgradients. Using the same ideas will enable us to improve

Lemma 1.1 to give a pointwise (rather than local) condition for a function to satisfy the LBSC.

If $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a lower semicontinuous function, we define $\operatorname{dom} f := \{x : f(x) < +\infty\}$. We say that ζ is a proximal subgradient of f at $x \in \operatorname{dom} f$, and we write $\zeta \in \partial_P f(x)$, if there exists $\eta > 0$ and $\sigma \geq 0$ such that

$$f(y) \geq f(x) + \langle \zeta, y - x\rangle - \sigma|y - x|^2 \quad \forall y \in B(x, \eta).$$

When f is convex, proximal subgradients coincide with convex subgradients.

We mention here some properties of proximal subgradients that will be used in the sequel (see [27] for proofs of these properties; the hypotheses stated here are far from being optimal).

First, consider the indicator function I_Γ of a $C^{1,1}$ hypersurface $\Gamma \subset \mathbb{R}^n$, that is $I_\Gamma(x) = 0$ if $x \in \Gamma$ and is $+\infty$ elsewhere. Then, for any $x \in \Gamma$, the set of proximal subgradients of I_Γ at x is the normal to the hypersurface Γ at x, $N_\Gamma(x)$.

If $f : U \to \mathbb{R}$ is a Lipschitz function on an open convex U, then f is convex if and only if for any $x, x' \in U$,

$$\langle \zeta - \zeta', x - x'\rangle \geq 0 \ \ , \ \forall \zeta \in \partial_P f(x) \ , \ \forall \zeta' \in \partial_P f(x').$$

When f is a Lipschitz function of Lipschitz rank K, then its proximal subgradients are bounded by K.

Nonsmooth analysis also provides several sum rules. Suppose that $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a lower semicontinuous function and $\theta : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$ function. If $\zeta \in \partial_P(f + \theta)(x)$, then $\zeta - \nabla\theta(x) \in \partial_P f(x)$.

If $\phi : \Gamma \to \mathbb{R}$ is a lower semicontinuous function, then we can extend it into a lower semicontinuous function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by setting $\phi(x) = +\infty$ whenever $x \notin \Gamma$. Then a proximal subgradient ζ of ϕ at $x \in \Gamma$ (we still denote $\zeta \in \partial_P\phi(x)$) will be defined as any proximal subgradient of $\tilde{\phi}$ at x, that is: there exist $\sigma \geq 0, \eta > 0$ such that

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x\rangle - \sigma|y - x|^2 \quad \forall y \in B(x, \eta) \cap \Gamma.$$

We consider a bounded open set Ω which is supposed to be of class $C^{1,1}$ and uniformly convex. Then, the tangent plane to Ω at any $x \in \Gamma$ is well-defined. For any $\zeta \in \partial_P\phi(x)$, we will denote by $\tilde{\zeta}$ the tangential component of ζ, that is the orthogonal projection of ζ on the tangent plane to Ω at x.

The main result of this section is

Theorem 1.7 *Suppose Ω is a bounded open uniformly convex set of class $C^{1,1}$ and consider $\phi : \Gamma \to \mathbb{R}$ a Lipschitz function of rank K. Then ϕ satisfies the LBSC if and only if for any $x \in \Gamma$, there is an open set U_x in $\mathbb{R}^n$ and some $Q \geq 0$ such that*

$$\langle \tilde{\zeta} - \tilde{\zeta}', y - y'\rangle \geq -Q|y - y'|^2, \tag{1.4}$$

for any $y, y' \in U_x \cap \Gamma$ and any $\zeta \in \partial_P\phi(y), \zeta' \in \partial_P\phi(y')$.

To prove Theorem 1.7, we are going to apply Theorem 1.2 for one implication and Proposition 1.2 for the other one. Suppose first that for any $x_0 \in \Gamma$, there is an open set U in $\mathbb{R}^n$ and some $Q \geq 0$ such that

$$\langle \tilde{\zeta} - \tilde{\zeta}', y - y'\rangle \geq -Q|y - y'|^2 \quad \forall y, y' \in U \cap \Gamma, \forall \zeta \in \partial_P\phi(y), \zeta' \in \partial_P\phi(y').$$

In view of the regularity of Γ, there exist open sets $U_1 \subset U, V \subset \mathbb{R}^{n-1}$ and $\rho : V \to \Gamma \cap U_1$ of class $C^{1,1}$ such that ρ is an immersion and a homeomorphism onto $\Gamma \cap U_1$. We can also suppose (see Claim 1.1) that there exists $\psi : U_1 \to V$ which is Lispchitz, $C^{1,1}$ and satisfies

$$\psi \circ \rho(v) = v \quad \forall v \in V, \quad \rho \circ \psi(x) = x \quad \forall x \in \Gamma \cap U_1.$$

Finally, shrinking V and U_1 if necessary, we can suppose that $\rho, D\rho$ are Lipschitz on $\mathrm{co}\, V$ and similarly $\psi, D\psi$ are Lipschitz on $\mathrm{co}\, U_1$. We will denote by R a Lipschitz constant for all these functions on these sets. Then to show that ϕ satisfies the LBSC, it is enough to prove that $\phi \circ \rho$ is linearly semiconvex. We need first to link the subgradients of ϕ to those of $\phi \circ \rho$, thanks to the following chain rule:

Lemma 1.2 *For any $v \in V$, $\xi \in \partial_P(\phi \circ \rho)(v)$, there exists $\zeta \in \partial_P \phi(\rho(v))$ such that $\xi = D\rho(v)^* \zeta = D\rho(v)^* \tilde{\zeta}$.*

Proof of Lemma 1.2: Let $v \in V, \xi \in \partial_P(\phi \circ \rho)(v)$. There exist $\eta > 0, \sigma \geq 0$ such that

$$\phi(\rho(v')) - \phi(\rho(v)) - \langle \xi, v' - v \rangle \geq -\sigma |v' - v|^2$$

for any $v' \in B(v, \eta)$. Denote $x' = \rho(v')$ and $x = \rho(v)$ so that $\psi(x') = v'$ and $\psi(x) = v$. There exists $F : U_1 \times U_1 \to \mathbb{R}^{n-1}$ uniformly bounded by R such that

$$\psi(y') - \psi(y) = D\psi(y)(y' - y) + F(y', y)|y' - y|^2$$

for any $y, y' \in U_1$. This implies

$$\begin{aligned}
\phi(x') - \phi(x) - \langle D\psi(x)^* \xi, x' - x \rangle &= \phi(x') - \phi(x) - \langle \xi, \psi(x') - \psi(x) \rangle \\
&\quad - |x' - x|^2 \langle \xi, F(x', x) \rangle \\
&= \phi(\rho(v')) - \phi(\rho(v)) - \langle \xi, v' - v \rangle \\
&\quad - |x' - x|^2 \langle \xi, F(x', x) \rangle \\
&\geq -\sigma |v' - v|^2 - |\xi||F(x', x)||x' - x|^2 \\
&\geq -R(\sigma + |\xi|)|x' - x|^2
\end{aligned}$$

since ψ is Lipschitz of rank R and F bounded by R. This inequality holds for any $x' \in \rho(B(v, \eta))$ which is a neighborhood of x in Γ. It follows that $D\psi(x)^* \xi \in \partial_P \phi(x)$. Let $\zeta := D\psi(x)^* \xi$. Then

$$D\rho(v)^* \zeta = D\rho(v)^* \circ D\psi(x)^* \xi = D(\psi \circ \rho)(v)^* \xi.$$

Moreover, for any $w \in V$, $\psi \circ \rho(w) = w$, hence $D(\psi \circ \rho)(v) = Id$ and so $D(\psi \circ \rho)(v)^* = Id$. We can conclude

$$D\rho(v)^* \zeta = \xi.$$

Finally, $D\rho(v)^* \zeta = D\rho(v)^* \tilde{\zeta}$, since the kernel of $D\rho(v)^*$ is exactly the normal $N_\Gamma(\rho(v))$ to the tangent plane to Ω at $\rho(v)$.

The lemma is proved.

□

We will also use the following well known result for semiconvex functions on open sets of $\mathbb{R}^{n-1}$.

Lemma 1.3 *Let $\theta : V \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function on an open set V. We suppose that there exists $Q \geq 0$ such that for any $v, v' \in V$ satisfying $[v, v'] \subset V$, we have*

$$\langle \xi' - \xi, v' - v \rangle \geq -Q|v' - v|^2 \quad \forall \xi \in \partial_P \theta(v), \xi' \in \partial_P \theta(v'). \tag{1.5}$$

Then θ is linearly semiconvex on V.

Proof : Consider the function $\theta_Q := \theta + Q/2|\cdot|^2$. Then on any convex subset of V, θ is linearly semiconvex if θ_Q is convex. Note also that $\zeta \in \partial_P \theta_Q(x)$ if and only if $\zeta - Qx \in \partial_P \theta(x)$. Hence, inequality (1.5) means

$$\langle \xi' - \xi, v' - v \rangle \geq 0 \ , \quad \forall \xi \in \partial_P \theta_Q(v), \ \xi' \in \partial_P \theta_Q(v'),$$

which implies the convexity of θ_Q on any convex subset of V. Hence θ is semiconvex on any convex subset of V, with the same modulus of semiconvexity $-Q|\cdot|$. This implies the semiconvexity of θ on V. Lemma 1.3 is proved.

□

We can now show that $\phi \circ \rho$ is linearly semiconvex. Let $v, v' \in V$ such that $[v, v'] \subset V$, $\xi' \in \partial_P \phi \circ \rho(v')$, $\xi \in \partial_P \phi \circ \rho(v)$. Let us estimate $< \xi' - \xi, v' - v >$. Thanks to Lemma 1.2, there exist $\zeta \in \partial_P \phi(\rho(v))$, $\zeta' \in \partial_P \phi(\rho(v'))$ such that

$$\xi = D\rho(v)^* \tilde{\zeta}, \qquad \xi' = D\rho(v')^* \tilde{\zeta}'.$$

There exists a function $E : V \times V \to \mathbb{R}^n$ bounded by R such that

$$\rho(w) - \rho(w') = D\rho(w')(w - w') + E(w, w')|w - w'|^2 \tag{1.6}$$

for any $w, w' \in V$. With $w' = v'$ and $w = v$, we get

$$\rho(v') - \rho(v) + E(v, v')|v - v'|^2 = D\rho(v')(v' - v).$$

With $w' = v$ and $w = v'$, we get

$$\rho(v') - \rho(v) - E(v', v)|v - v'|^2 = D\rho(v)(v' - v).$$

Thus, using (1.4),

$$\begin{array}{rcl}
\langle \xi' - \xi, v' - v \rangle & = & \langle D\rho(v')^* \tilde{\zeta}' - D\rho(v)^* \tilde{\zeta}, v' - v \rangle \\
& = & \langle \tilde{\zeta}', D\rho(v')(v' - v) \rangle - \langle \tilde{\zeta}, D\rho(v)(v' - v) \rangle \\
& = & \langle \tilde{\zeta}', \rho(v') - \rho(v) \rangle - \langle \tilde{\zeta}, \rho(v') - \rho(v) \rangle \\
& & + \langle \tilde{\zeta}', E(v, v')|v - v'|^2 \rangle + \langle \tilde{\zeta}, E(v', v)|v - v'|^2 \rangle \\
& \geq & -Q|\rho(v') - \rho(v)|^2 - 2RK|v - v'|^2 \\
& \geq & -(QR^2 - 2RK)|v' - v|^2
\end{array}$$

(Q is given by (1.4), R is a Lipschitz constant for ρ on V, K is a Lipschitz constant for ϕ on Γ. Finally, E is bounded by R on $V \times V$.)

Apply now Lemma 1.3 to conclude that $\phi \circ \rho$ is linearly semiconvex on V, and so ϕ restricted to $\rho(V)$ satisfies the LBSC. As in the proof of Theorem 1.2, we infer from this fact that ϕ satisfies the LBSC.

Let us now prove the converse. We could reverse the arguments of the first part of the proof but here is a different strategy. Suppose that ϕ satisfies the LBSC. Then ϕ is the restriction to Γ of a convex function $\bar{\phi} : \mathbb{R}^n \to \mathbb{R}$. Let I_Γ be the indicator function of Γ. Let $\tilde{\phi} := \bar{\phi} + I_\Gamma$. Then for any $x \in \Gamma$, $\partial_P \phi(x) = \partial_P \tilde{\phi}(x)$ (by definition of $\partial_P \phi(x)$).

The *limiting sum rule* (see [27], Proposition 10.1) shows that for any $x \in \Gamma$, $\zeta \in \partial_P \phi(x)$, there exist $\nu \in N_\Gamma(x)$, $\lambda \in \partial \bar{\phi}(x)$ (recall that for a convex function, proximal subgradients are convex subgradients) such that

$$\zeta = \nu + \lambda.$$

Considering the orthogonal projection of this equality on the tangent hyperplane to Γ at x, we have:

$$\tilde{\zeta} = \tilde{\lambda}.$$

Hence, to show that inequality (1.4) holds, it is enough to show that for any $x \in \Gamma$, there exist an open set U_x in $\mathbb{R}^n$ and some $Q \geq 0$ such that

$$\langle \tilde{\lambda} - \tilde{\lambda}', y - y' \rangle \geq -Q|y - y'|^2 \tag{1.7}$$

for any $y, y' \in \Gamma \cap U_x$, and $\lambda \in \partial \bar{\phi}(y), \lambda' \in \partial \bar{\phi}(y')$. For any $\lambda \in \partial \bar{\phi}(x)$, note that $\lambda - \tilde{\lambda} \in N_\Gamma(x)$. Then, inequality (1.7) is an easy consequence of the following:

Lemma 1.4 *For any $x \in \Gamma$, there is an open set U_x in $\mathbb{R}^n$ and some $Q_0 \geq 0$ such that for any y, $y' \in \Gamma \cap U_x$, and any $\nu \in N_\Gamma(y)$, $\nu' \in N_\Gamma(y')$, we have*

$$\langle \nu - \nu', y - y' \rangle \leq Q_0(|\nu| + |\nu'|)|y - y'|^2.$$

Suppose that Lemma 1.4 is true. Then, let $y \in \Gamma$ and U_x, Q_0 as in the lemma. For any $y, y' \in \Gamma \cap U_x$, and $\lambda \in \partial \bar{\phi}(y)$, $\lambda' \in \partial \bar{\phi}(y')$, we have (with $\nu = \lambda - \tilde{\lambda}$, $\nu' = \lambda' - \tilde{\lambda}'$)

$$\begin{array}{rcl} \langle \tilde{\lambda} - \tilde{\lambda}', y - y' \rangle & \geq & \langle \lambda - \lambda', y - y' \rangle - \langle \nu - \nu', y - y' \rangle \\ & \geq & 0 - Q_0(|\nu| + |\nu'|)|y - y'|^2 \text{ (because } \bar{\phi} \text{ is convex)} \\ & \geq & -Q|y - y'|^2. \end{array}$$

The last line follows from the fact that $\bar{\phi}$ is Lipschitz on a neighborhood of Γ, which implies that its convex subgradients are locally bounded and so are the normal components of these. Then inequality (1.7) holds provided that we show Lemma 1.4.

Let $x \in \Gamma$ and $\rho : V \to U_x$ be a parametrization near x as in the first part of the proof of Theorem 1.7. Then for any $y = \rho(v)$, $y' = \rho(v')$, $\nu \in N_\Gamma(y)$, $\nu' \in N_\Gamma(y')$, we have:

$$\begin{array}{rcl} \langle \nu - \nu', y - y' \rangle & = & \langle \nu - \nu', \rho(v) - \rho(v') \rangle \\ & = & \langle \nu, \rho(v) - \rho(v') \rangle - \langle \nu', \rho(v) - \rho(v') \rangle \\ & \leq & \langle \nu, D\rho(v)(v - v') \rangle - \langle \nu', D\rho(v')(v - v') \rangle \\ & & \quad +R(|\nu| + |\nu'|)|v - v'|^2 \quad \text{(thanks to} \quad (1.6)\,) \\ & \leq & R(|\nu| + |\nu'|)|v - v'|^2 \end{array}$$

since ν is in the kernel of $D\rho(v)^*$ and the same is true for $\nu', D\rho(v')^*$.

Finally, using the fact that $\psi_{|\Gamma} = \rho^{-1}$ is Lipschitz of rank R on U_x, we find

$$\langle \nu - \nu', y - y' \rangle \leq R^2(|\nu| + |\nu'|)|y - y'|^2,$$

which is the desired estimate with $Q_0 = R^2$.

□

The developments above lead to the following result, which significantly improves Lemma 1.1, as it shows that merely a pointwise condition guarantees the LBSC.

Proposition 1.6 *Let Ω be a uniformly convex bounded open set of class $C^{1,1}$. We suppose that $\phi : \Gamma \to \mathbb{R}$ is continuous and there exists $Q \geq 0$ such that for any $x \in \Gamma$, there exists $\zeta \in \bar{B}(0, Q)$ satisfying*

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle$$

for any $y \in \Gamma$ near x. Then ϕ satisfies the LBSC.

Proof: It is enough to show that if $\rho : V \to \Gamma$ is a parametrization as in the proof of Theorem 1.7, then $\phi \circ \rho$ is linearly semiconvex. For any $v \in V$, there exists $\zeta \in \bar{B}(0, Q)$ such that

$$\phi \circ \rho(v') - \phi \circ \rho(v) \geq \langle \zeta, \rho(v') - \rho(v) \rangle$$

for any v' near v. Set $\xi := D\rho(v)^* \zeta$. We have

$$\phi \circ \rho(v') - \phi \circ \rho(v) \geq \langle \xi, v' - v \rangle - \sigma |v' - v|^2$$

for any v' near v, (σ does depend only on Q and on the modulus of continuity of $D\rho$). Set $\theta : V \to \mathbb{R}$, $\theta = \phi \circ \rho$. Then θ satisfies the hypotheses of the following lemma.

Lemma 1.5 *Let $\theta : V \to \mathbb{R}$ be a continuous function. We suppose there exists $\sigma \geq 0$ such that for any $v \in V$, there exists $\xi \in \mathbb{R}^n$, satisfying*

$$\theta(v') \geq \theta(v) + \langle \xi, v' - v \rangle - \sigma |v' - v|^2$$

for any v' near v. Then θ is linearly semiconvex on V.

Lemma 1.5 concludes the proof of the proposition. Let us now prove it. Set $g := \theta + \sigma |\cdot|^2$. Then for any $v \in V$, there exists $\xi \in \mathbb{R}^n$ such that

$$g(v') \geq g(v) + \langle \xi + 2\sigma v, v' - v \rangle$$

for any v' near v, so that g is convex. Then $\theta = g - \sigma |\cdot|^2$ is linearly semiconvex, and Lemma 1.5 is proved.

□

In Proposition 1.6, the continuity assumption is necessary in view of the following example: Ω is the unit disc in $\mathbb{R}^2$ and $\phi : (\cos\theta, \sin\theta) \mapsto \theta \in [0, 2\pi[$. Furthermore, the existence of some *a priori* rank Q is unavoidable, as shown by the following example; here, Ω is the unit disc in $\mathbb{R}^2$ and Γ the unit circle.

Example 1.2 *There exists $\phi \in C^1(\Gamma)$ such that for any $x \in \Gamma$, there exists some $\zeta \in \mathbb{R}^n$ satisfying*

$$\phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle \quad \forall y \in \Gamma, \tag{1.8}$$

and yet ϕ does not satisfy the LBSC.

Proof : There exists $g \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}-2\pi\mathbb{Z})$, 2π periodic, nonnegative, which is equal to $g(\theta) := |\theta|^3(sin\frac{1}{\theta}+1)$ on a neighbourhood of 0 when $\theta \neq 0$ and vanishes at 0. Set $\phi(cos\theta, sin\theta) := g(\theta)$. The map g is nonnegative on a neighbourhood of 0 hence $(1,0)$ is a global minimum of ϕ. Therefore, $\phi(y) \geq \phi(1,0) + < (0,0), y-(1,0) >$ for any $y \in \Gamma$. On $\Gamma - \{(1,0)\}$, ϕ is C^2 so that ϕ restricted to $\Gamma \cap]-\infty, 1[\times\mathbb{R}$ satisfies the LBSC (Ω being uniformly convex). To sum up, ϕ satisfies (1.8). Let us show that ϕ does not satisfy the LBSC. Let $x = (\cos\theta, \sin\theta) \in \Gamma$ near $(1,0)$ with $\theta > 0$ and $\zeta \in \mathbb{R}^n$ such that (1.8) holds. Then the tangential component of ζ is

$$\tilde{\zeta} = g'(\theta) = 3\theta^2(\sin\frac{1}{\theta}+1) - \theta\cos\frac{1}{\theta}$$

and the normal component must satisfy (as a direct consequence of (1.8))

$$\hat{\zeta} \geq \frac{g(\theta') - g(\theta) - g'(\theta)\sin(\theta'-\theta)}{\cos(\theta'-\theta)-1}$$

for any $\theta' \in \mathbb{R}$. When $\theta > 0$, the right hand side tends to $-g''(\theta) = -6\theta(\sin 1/\theta + 1) + 1/\theta \sin 1/\theta + 4\cos 1/\theta$ when $\theta' \to \theta$. Since $-g''(\frac{2}{(4n+1)\pi}) \to +\infty$ when $n \to +\infty$, we infer that $\hat{\zeta}$ cannot be majorized, hence ϕ does not satisfy the LBSC.

Chapter 2

Local Lipschitz continuity of a problem in the Calculus of Variations

This chapter is based on the paper *Local Lipschitz continuity of solutions to a problem in the calculus of variations* (with F. Clarke, submitted).

2.1 Introduction

We study the regularity of solutions to the following problem (P) in the multiple integral calculus of variations:

$$\min_u \int_\Omega \{F(Du(x)) + G(x, u(x))\}\, dx \text{ subject to } u \in W^{1,1}(\Omega),\ \operatorname{tr} u = \phi,$$

where Ω is a domain in $\mathbb{R}^n$, u is scalar-valued, and $\operatorname{tr} u$ signifies the trace of u on $\Gamma := \partial\Omega$.

The aim is to deduce local Lipschitz regularity from properties of the boundary function ϕ. This is in the general spirit of the well-known Hilbert-Haar theory (see for example [37], [71]), which requires that ϕ satisfy the *bounded slope condition* (BSC). The BSC of rank K is the assumption that, given any point $\gamma \in \Gamma$, there exist two affine functions

$$y \mapsto \langle \zeta_\gamma^-, y - \gamma \rangle + \phi(\gamma),\ y \mapsto \langle \zeta_\gamma^+, y - \gamma \rangle + \phi(\gamma)$$

agreeing with ϕ at γ, whose slopes satisfy $|\zeta_\gamma^-| \leq K$, $|\zeta_\gamma^+| \leq K$, and such that

$$\langle \zeta_\gamma^-, \gamma' - \gamma \rangle + \phi(\gamma) \leq \phi(\gamma') \leq \langle \zeta_\gamma^+, \gamma' - \gamma \rangle + \phi(\gamma) \quad \forall \gamma' \in \Gamma.$$

The classical Hilbert-Haar theorem asserts that if F is convex, $G = 0$, and ϕ satisfies the BSC, then there exists a (globally) Lipschitz minimizer for (P). The first proof of this statement is due to Miranda [69], although there are several special cases that are antecedents to this. The case in which G is different from 0 has been treated by Stampacchia [84] (and implicitly in [47]) under stronger smoothness assumptions on the data than used here; see also Cellina [24] for more recent developments involving the BSC.

The BSC is a restrictive requirement on flat parts of Γ, since it forces ϕ to be affine. Moreover, if Ω is smooth, then it forces ϕ to be smooth as well (see Hartman [43] for precise statements). Recently, Clarke [28] has introduced a new hypothesis on ϕ, the *lower bounded slope condition* (LBSC) of rank K : given any point γ on the boundary, there exists an affine function

$$y \mapsto \langle \zeta_\gamma, y - \gamma \rangle + \phi(\gamma)$$

with $|\zeta_\gamma| \leq K$ such that

$$\langle \zeta_\gamma, \gamma' - \gamma \rangle + \phi(\gamma) \leq \phi(\gamma') \quad \forall \gamma' \in \Gamma.$$

This requirement, which can be viewed as a one-sided BSC, enlarges considerably the class of boundary functions which it allows (compared to the BSC). The property has been studied by Bousquet in [12], where it is shown that ϕ satisfies the LBSC if and only if it is the restriction to Γ of a convex function. When Ω is uniformly convex, ϕ satisfies the LBSC if and only if it is the restriction to Γ of a semiconvex function.

It turns out that the LBSC has significant implications for the regularity of the solution u, although it implies less than the full, two-sided BSC. In fact, it is shown in [28] that in the case where $G = 0$, the one-sided BSC gives the crucial regularity property that one seeks: u is *locally* Lipschitz in Ω. This allows one to assert that u is a weak solution of the Euler equation, in the absence of the usual upper growth conditions on F. Furthermore, the local Lipschitz property allows one to invoke De Giorgi's regularity theory (when the data are sufficiently smooth) to obtain the continuous differentiability of the solution.

The goal of this article is to prove local Lipschitz regularity of the solution for a class of problems with G different from 0, under weak regularity hypotheses on the data of the problem, and when the LBSC is satisfied (rather than the BSC). The next section describes the hypotheses and gives a self-contained proof of the main theorem of the article. It is most closely related to the work of Stampacchia, but the method of proof differs in several important respects. A variant of the main theorem is developed in Section 3, and the final section discusses the issue of the continuity of the solution at the boundary.

2.2 The main result

We now specify the hypotheses on the data of the problem (P). The first one, in particular, justifies the use of trace.

$(H\Omega)$ Ω is an open bounded convex set.

We require that F be uniformly elliptic, and that G be locally Lipschitz in u. More precisely:

(HF) For some $\mu > 0$, F satisfies, for all $\theta \in (0,1)$ and $p, q \in \mathbb{R}^n$:

$$\theta F(p) + (1-\theta)F(q) \geq F(\theta p + (1-\theta)q) + (\mu/2)\theta(1-\theta)|p-q|^2.$$

We remark that when F is of class C^2, (HF) holds if and only if, for every $v \in \mathbb{R}^n$, we have

$$\langle z, \nabla^2 F(v) z \rangle \geq \mu |z|^2 \quad \forall\, z \in \mathbb{R}^n.$$

Under (HF), it is easy to see that $\int_\Omega F(Dw)\, dx$ is well-defined (possibly as $+\infty$) for any $w \in W^{1,1}(\Omega)$.

(HG) $G(x,u)$ is measurable in x and differentiable in u, and for every bounded interval U in $\mathbb{R}$, there is a constant L such that for almost all $x \in \Omega$,

$$|G(x,u) - G(x,u')| \leq L|u - u'| \quad \forall\, u, u' \in U.$$

We also postulate as part of (HG) that for some bounded function b, the integral $\int_\Omega G(x, b(x))\, dx$ is well-defined and finite. It follows that the same is true for all bounded measurable functions w.

In the presence of $(H\Omega), (HF)$, and (HG), it follows that

$$I(w) := \int_\Omega \{F(Dw(x)) + G(x, w(x))\}\, dx$$

is well-defined for all $w \in W^{1,1}(\Omega)$ for which w is bounded. We say that u solves (P) relative to $L^\infty(\Omega)$ if u is itself bounded, and if we have $I(u) \leq I(w)$ for all bounded w that are admissible for (P).

The theorem to be proved is the following.

Theorem 2.1
Under the hypotheses $(H\Omega), (HF)$, and (HG), and when ϕ satisfies the Lower Bounded Slope Condition, any solution u of (P) relative to $L^\infty(\Omega)$ is locally Lipschitz in Ω.

In the context of the theorem, even when $G = 0$ and $F(v) = |v|^2$, a bounded solution u of (P) may fail to be globally Lipschitz; an example of this type is given in [12], [28]. Let us also point out that the theorem has an alternate version in which the LBSC is replaced by the *upper* BSC; the conclusion is the same. Finally, we remark that Stampacchia [84] has described structural assumptions on G which guarantee *a priori* the existence and boundedness of solutions of (P); these will be described in the next section.

2.2.1 The lower barrier condition

The proof of the main result uses in part the well-known barrier technique. Our one-sided version of this is the following.

Theorem 2.2
Under hypotheses $(H\Omega), (HF)$, and (HG), let u be a bounded solution of problem (P) as described above, where ϕ satisfies the Lower Bounded Slope Condition of rank K. Then there exists $\bar{K} > 0$ with the following property: for any $\gamma \in \Gamma$ there exists a function w which is Lipschitz of rank $\bar{K}$, which agrees with ϕ at γ, and which satisfies $w \leq u$ a.e. in Ω.

Proof As observed in the introduction, we may suppose that ϕ is a globally defined convex function of Lipschitz rank K. Thus there is an element ζ with $|\zeta| \leq K$ in the subdifferential of ϕ at γ:

$$\phi(x) - \phi(\gamma) \geq \langle \zeta, x - \gamma \rangle \quad \forall\, x \in \mathbb{R}^n.$$

By (HG) there is a Lipschitz constant L valid for $G(x, \cdot)$ over the interval

$$\left[-\|u\|_{L^\infty(\Omega)}, \|\phi\|_{L^\infty(\Gamma)} + K \operatorname{diam} \Omega \right],$$

for $x \in \Omega$ a.e. Fix any $T > (L+1)\exp(\operatorname{diam}\Omega)/\mu$, where μ is given by (HF).

The following construction is a refinement of that proposed by Hartmann and Stampacchia [47] (Lemma 10.1). Let ν be a unit outward normal vector to $\overline{\Omega}$ at γ, and define

$$w(x) := \phi(\gamma) + \langle \zeta, x - \gamma \rangle - T\{1 - \exp(\langle x - \gamma, \nu \rangle)\}.$$

We proceed to prove that w has the required properties. Clearly w agrees with ϕ at γ, and is Lipschitz of rank

$$\bar{K} := K + T \exp(\operatorname{diam}\Omega).$$

We need only show that the set

$$S := \{x \in \Omega : w(x) > u(x)\}$$

has measure 0.

The function $M(x) := \max[u(x), w(x)]$ belongs to $W^{1,1}(\Omega)$ (see for example [36] or [62]), and we have:

$$DM(x) = Dw(x), x \in S \text{ a.e.}, \; DM(x) = Du(x), x \in \Omega \backslash S \text{ a.e.}$$

It follows from the subgradient inequality for ζ that $M \in \phi + W_0^{1,1}(\Omega)$ (in deriving this, we also use the fact that $\langle x - \gamma, \nu \rangle \leq 0$ for $x \in \Omega$). By the optimality of u (relative to M) we deduce

$$\int_S \{F(Du(x)) + G(x, u(x))\}\, dx \leq \int_S \{F(Dw(x)) + G(x, w(x))\}\, dx.$$

The Lipschitz condition satisfied by G now leads to

$$\int_S \{F(Du(x)) - F(Dw(x))\}\, dx \leq L \int_S \{w(x) - u(x)\}\, dx. \tag{2.1}$$

In deriving the next estimate (which concludes the proof), let us make the temporary assumption that F is smooth (C^2 or better). Then, by straightforward calculation, the function $\psi(x) := \nabla F(Dw(x))$ satisfies

$$\operatorname{div} \psi(x) = T \exp(\langle x - \gamma, \nu \rangle) \left\langle \nu, \nabla^2 F(Dw(x)) \nu \right\rangle \geq L + 1, \tag{2.2}$$

in light of (HF), and because of how T was chosen. We proceed to deduce from (2.1) the following:

$$\begin{aligned} L \int_S \{w(x) - u(x)\}\, dx &\geq \int_S \{F(Du(x)) - F(Dw(x))\}\, dx \\ &\geq \int_S \langle \psi(x), Du(x) - Dw(x) \rangle \, dx \end{aligned}$$

(by the subdifferential inequality)

$$= \int_\Omega \langle \psi(x), D\min[u,w](x) - Dw(x)\rangle \, dx$$
$$= \int_\Omega (\operatorname{div}\psi(x))(w(x) - \min[u,w](x))\, dx$$

(integration by parts, noting that $\min[u,w] = w$ on Γ)

$$\geq (L+1)\int_\Omega \{w(x) - \min[u,w](x)\}\, dx$$

(in view of (2.2))

$$\geq (L+1)\int_S \{w(x) - u(x)\}\, dx.$$

This shows that S is of measure 0, since $w - u > 0$ in S.

In the general case in which F is not smooth, we consider a nondecreasing sequence $\{F_k\}_{k\in\mathbb{N}}$ of functions in $C^\infty(\mathbb{R}^n)$ converging to F uniformly on bounded sets, and such that the ellipticity condition in (HF) holds for F_k when p, q are restricted to a ball $\overline{B}(0, \overline{K}+1)$ containing all the values of Dw. Such a sequence exists by a mollification-truncation argument; see Morrey [71], Lemma 4.2.1. Then, arguing as above, we derive, for any $k \geq 1$,

$$\int_S \{F_k(Du(x)) - F_k(Dw(x))\}\, dx \geq (L+1)\int_S \{w(x) - u(x)\}\, dx.$$

The result now follows from the Monotone Convergence Theorem. □

2.2.2 Proof of Theorem 2.1

Let λ and q be parameters satisfying

$$\lambda \in [1/2, 1),\ q > \bar{q} := \bar{K}\operatorname{diam}\Omega + ||\phi||_{L^\infty(\Gamma)},$$

and fix any point $z \in \Gamma$. We denote

$$\Omega_\lambda := \lambda(\Omega - z) + z.$$

Note that Ω_λ is a subset of Ω, since the latter is convex. We proceed to define the following function on Ω_λ:

$$u_\lambda(x) := \lambda u\left((x-z)/\lambda + z\right) - q(1-\lambda).$$

Then u_λ belongs to $W_0^{1,1}(\Omega_\lambda) + \phi_\lambda$, where

$$\phi_\lambda(y) := \lambda\phi((y-z)/\lambda + z) - q(1-\lambda).$$

For every $x \in \mathbb{R}^n$, we will denote $(x-z)/\lambda + z$ by x_λ.

We are now going to compare u_λ and u on $\Gamma_\lambda := \partial\Omega_\lambda$; this comparison via dilation was introduced in [28].

Lemma 1 We have $u_\lambda \leq u$ on Γ_λ.

The meaning of this inequality is that $(u_\lambda - u)^+ := \max(0, u_\lambda - u)$ belongs to $W_0^{1,1}(\Omega_\lambda)$, where here u signifies of course the restriction of u to Ω_λ. To prove the lemma, recall first that in the preceding section we proved the existence, for any $\gamma \in \Gamma$, of a $\bar{K}$-Lipschitz function w_γ such that $w_\gamma(\gamma) = \phi(\gamma)$ and $w_\gamma \leq u$ a.e. in Ω (which implies $w_\gamma \leq \phi$ on Γ).

Introduce $l(y) := \sup_{\gamma\in\Gamma} w_\gamma(y)$. Then l is a $\bar{K}$-Lipschitz function which coincides with ϕ on Γ and which has $l \leq u$ a.e. on Ω. Thus $u - l \in W_0^{1,1}(\Omega)$. There exists therefore a sequence $v_m \in \mathrm{Lip}_0(\Omega)$ converging to $u - l$ in $W^{1,1}(\Omega)$ and almost everywhere in Ω. We can suppose moreover $v_m \geq 0$, by replacing v_m by $v_m^+ := \max(v_m, 0)$. We have used here the fact that if a sequence of functions k_m converges almost everywhere and in $W^{1,1}(\Omega)$ to k, then k_m^+ converges to k^+ in $W^{1,1}(\Omega)$.

We define the functions

$$u_m(x) := v_m(x) + l(x) \ , \ \ u_{m\lambda}(x) := \lambda u_m((x-z)/\lambda + z) - q(1-\lambda).$$

These regularizations of u and u_λ will allow us to complete the proof of the lemma.

We have $u_m \in C^0(\bar{\Omega})$, $l \leq u_m$ on Ω and $u_m = \phi = l$ on Γ. We claim that $u_{m\lambda}(\gamma) \leq u_m(\gamma)$ for every $m \geq 0, \gamma \in \Gamma_\lambda$. Suppose for a moment this claim were true. Then we could assert that

$$(u_{m\lambda} - u_m)^+ \in W_0^{1,1}(\Omega_\lambda).$$

Now, $u_{m\lambda}$ tends to u_λ in $W^{1,1}(\Omega_\lambda)$ and almost everywhere, as does u_m to u. It would follow therefore that $(u_\lambda - u)^+ \in W_0^{1,1}(\Omega_\lambda)$, which is what we wish to prove.

So it suffices to prove the claim. Fix some $\gamma \in \Gamma_\lambda$. Then,

$$\begin{aligned}
u_{m\lambda}(\gamma) - u_m(\gamma) &= \lambda u_m(\gamma_\lambda) - u_m(\gamma) - q(1-\lambda) \\
&\leq \lambda\phi(\gamma_\lambda) - l(\gamma) - q(1-\lambda) \\
&= \lambda l(\gamma_\lambda) - l(\gamma) - q(1-\lambda) \\
&\leq (l(\gamma_\lambda) - l(\gamma)) + (1-\lambda)(||l||_{L^\infty(\Gamma)} - q) \\
&\leq \bar{K}|\gamma - \gamma_\lambda| + (1-\lambda)(||l||_{L^\infty(\Gamma)} - q) \\
&\leq (1-\lambda)(\bar{K}\,\mathrm{diam}\,\Omega + ||l||_{L^\infty(\Gamma)} - q) \\
&\leq 0
\end{aligned}$$

in light of the way q has been chosen. This proves the claim and completes the proof of Lemma 1.

□

The next step of the proof is to show that the set

$$A := \{y \in \Omega_\lambda : u_\lambda(y) > u(y)\}$$

has measure zero. Let $w(x) := \min(u, u_\lambda)$, which belongs to $W_0^{1,1}(\Omega_\lambda) + \phi_\lambda$ in light of Lemma 1, and define

$$w^\lambda(x) := \frac{1}{\lambda} w(\lambda(x-z)+z) + q(\frac{1}{\lambda} - 1),$$

an element of $W_0^{1,1}(\Omega)+\phi$. Fix any $\theta\in(0,1)$. Then $v:=\theta w^\lambda+(1-\theta)u$ lies in $W_0^{1,1}(\Omega)+\phi$, so that $I(u)\le I(v)$, which yields after an evident change of variables

$$\begin{aligned}
&\int_{\Omega_\lambda}\Big\{F(Du_\lambda)+G(\frac{y-z}{\lambda}+z,\frac{u_\lambda+q(1-\lambda)}{\lambda})\Big\}\,dy\le\\
&\qquad\int_{\Omega_\lambda}\Big\{F(\theta Dw+(1-\theta)Du_\lambda)+G(\frac{y-z}{\lambda}+z,\theta\frac{w+q(1-\lambda)}{\lambda}+\\
&\qquad\qquad(1-\theta)\frac{u_\lambda+q(1-\lambda)}{\lambda})\Big\}\,dy.
\end{aligned}$$

We also note that the right side is finite, since $\int_\Omega F(Du)\,dx$ is finite, and in light of the convexity of F. This implies

$$\begin{aligned}
&\int_{A}\Big\{F(Du_\lambda)+G(\frac{y-z}{\lambda}+z,\frac{u_\lambda+q(1-\lambda)}{\lambda})\Big\}\,dy\le\\
&\qquad\int_{A}\Big\{F(\theta Dw+(1-\theta)Du_\lambda)+G(\frac{y-z}{\lambda}+z,\theta\frac{w+q(1-\lambda)}{\lambda}+\\
&\qquad\qquad(1-\theta)\frac{u_\lambda+q(1-\lambda)}{\lambda})\Big\}\,dy,
\end{aligned}$$

whence (since $w=u$ on A)

$$\begin{aligned}
&\int_{A}\{F(Du_\lambda)-F(\theta Du+(1-\theta)Du_\lambda)\}\,dy\le\\
&\qquad\int_{A}\Big\{G(\frac{y-z}{\lambda}+z,\theta\frac{u+q(1-\lambda)}{\lambda}+(1-\theta)\frac{u_\lambda+q(1-\lambda)}{\lambda})\\
&\qquad\qquad-G(\frac{y-z}{\lambda}+z,\frac{u_\lambda+q(1-\lambda)}{\lambda})\Big\}\,dy. \qquad (2.3)
\end{aligned}$$

Now let $W(x):=\max(u(x),u_\lambda(x))$ for $x\in\Omega_\lambda$, and $W(x):=u(x)$ for $x\in\Omega\backslash\Omega_\lambda$. Then $W\in W_0^{1,1}(\Omega)+\phi$ since $u_\lambda\le u$ on Γ_λ. With $v:=\theta W+(1-\theta)u$, we have $I(u)\le I(v)$, which yields

$$\begin{aligned}
&\int_A\{(F(Du)+G(y,u))\}\,dy\le\\
&\qquad\int_A\{(F(\theta Du_\lambda+(1-\theta)Du)+G(y,\theta u_\lambda+(1-\theta)u))\}\,dy,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_A\{(F(Du)-F(\theta Du_\lambda+(1-\theta)Du))\}\,dy\le\\
&\qquad\int_A\{(G(y,\theta u_\lambda+(1-\theta)u)-G(y,u))\}\,dy. \qquad (2.4)
\end{aligned}$$

Summing (2.3) and (2.4), we get

$$\begin{aligned}
&\int_A \Big\{(1-\theta)F(Du_\lambda) + \theta F(Du) - F(\theta Du + (1-\theta)Du_\lambda) \\
&\qquad + \theta F(Du_\lambda) + (1-\theta)F(Du) - F(\theta Du_\lambda + (1-\theta)Du)\Big\}\, dy \\
&\qquad \le \int_A \Big\{G(\frac{y-z}{\lambda} + z, \theta\frac{u+q(1-\lambda)}{\lambda} + (1-\theta)\frac{u_\lambda + q(1-\lambda)}{\lambda}) \\
&\qquad\qquad - G(\frac{y-z}{\lambda} + z, \frac{u_\lambda + q(1-\lambda)}{\lambda}) \\
&\qquad\qquad\qquad + G(y, \theta u_\lambda + (1-\theta)u) - G(y,u)\Big\}\, dy. \quad (2.5)
\end{aligned}$$

Thanks to (HF) we see that the left side of the last inequality is no less than

$$\mu\theta(1-\theta)\int_A |Du_\lambda - Du|^2 dy.$$

Substituting into (2.5), dividing by θ, and letting θ go to 0, we find

$$\mu \int_A |Du_\lambda - Du|^2\, dy \le \int_A (\frac{1}{\lambda}g(\frac{y-z}{\lambda} + z) - g(y))(u_\lambda - u)\, dy,$$

where we have denoted by $g(y)$ the function $-G_u(y, u(y))$, which belongs to $L^\infty(\Omega)$. Write for any $y \in \Omega_\lambda$:

$$\frac{1}{\lambda}g(\frac{y-z}{\lambda} + z) - g(y) = (\frac{1}{\lambda} - \frac{1}{\lambda^n})g(\frac{y-z}{\lambda} + z) + h(y)$$

where we define $h(y) := 1/\lambda^n g((y-z)/\lambda + z) - g(y)$ for $y \in \Omega_\lambda$ and $h(y) = 0$ for $y \in \mathbb{R}^n \backslash \Omega_\lambda$. Then

$$\mu \int_A |Du_\lambda - Du|^2 dy \le \int_A [(\frac{1}{\lambda^n} - \frac{1}{\lambda})g_0 + h(y)](u_\lambda - u)\, dy, \tag{2.6}$$

where $g_0 := ||g||_\infty$.

Lemma 2 There exists $f = (f_1, \ldots, f_n) \in L^\infty(\mathbb{R}^n)^n$ such that for each $j \in \{1, \ldots, n\}$, for almost every $(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \in \mathbb{R}^{n-1}$, the function

$$y_j \mapsto f_j(y_1, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_n)$$

is absolutely continuous on $\mathbb{R}$ with

$$\frac{\partial f_j}{\partial y_j} \in L^\infty(\mathbb{R}^n)$$

and

$$\operatorname{div} f := \sum_{j=1}^n \frac{\partial f_j}{\partial y_j} = h \text{ a.e. in } \Omega_\lambda \tag{2.7}$$

and such that $||f||_{L^\infty(\Omega)} \le C_0(1-\lambda)$ for some constant C_0 which depends only on g_0 and Ω (λ being restricted to $[1/2, 1)$).

Proof of Lemma 2 We extend g by setting it equal to 0 outside Ω. There exists $(c_1,\dots,c_n)\in\mathbb{R}^n$ such that

$$\Omega\subset\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_j\ge c_j\ \forall j=1,\dots,n\}.$$

Define $f_1(y_1,\dots,y_n)$ to be

$$\int_{c_1}^{y_1}\left[\frac{1}{\lambda}g(\frac{y_1'-z_1}{\lambda}+z_1,y_2,\dots,y_n)-g(y_1',y_2,\dots,y_n)\right]dy_1'$$

and similarly set $f_j(y_1,\dots,y_n)$ equal to

$$\int_{c_j}^{y_j}\Big[\frac{1}{\lambda^j}g(\frac{y_1-z_1}{\lambda}+z_1,\dots,\frac{y_j'-z_j}{\lambda}+z_j,y_{j+1},\dots,y_n)-$$
$$\frac{1}{\lambda^{j-1}}g(\frac{y_1-z_1}{\lambda}+z_1,\dots,\frac{y_{j-1}-z_{j-1}}{\lambda}+z_{j-1},y_j',y_{j+1},\dots,y_n)\Big]\,dy_j'.$$

This implies

$$\frac{\partial f_j}{\partial y_j}\in L^\infty(\mathbb{R}^n)$$

and that f satisfies (2.7). Upon making the change of variables $y_j''=(y_j'-z_j)/\lambda+z_j$ in the first integral defining f_j, we find readily

$$\begin{aligned}|f_j(y_1,...,y_n)|&\le g_0\frac{1}{\lambda^{j-1}}\{|\frac{y_j-z_j}{\lambda}+z_j-y_j|+|\frac{c_j-z_j}{\lambda}+z_j-c_j|\}\\&=g_0\frac{1-\lambda}{\lambda^j}\{|y_j-z_j|+|c_j-z_j|\}\\&\le g_0\frac{1-\lambda}{\lambda^j}\{\operatorname{diam}\Omega+|c|+\operatorname{dist}(0,\Omega)\},\end{aligned}$$

if the y_i lie in Ω. This completes the proof of the lemma.

Thanks to Lemma 2, we can write (2.6) as

$$\mu\int_A|Du_\lambda-Du|^2dy\le$$
$$\int_A\left\{\langle f,Du-Du_\lambda\rangle+g_0(\frac{1}{\lambda^n}-\frac{1}{\lambda})(u_\lambda-u)\right\}dy.\quad(2.8)$$

Now Poincaré's inequality, applied to $(u_\lambda-u)^+\in W_0^{1,1}(\Omega_\lambda)$, yields the existence of a constant C_P which depends only on Ω such that

$$\int_A(u_\lambda-u)\,dy\le C_P\int_A|Du_\lambda-Du|\,dy.$$

Then (2.8) implies

$$\mu\int_A|Du_\lambda-Du|^2dy\le\Big[||f||_{L^\infty(\Omega)}+g_0C_P(\frac{1}{\lambda^n}-\frac{1}{\lambda})\Big]\int_A|Du_\lambda-Du|\,dy.$$

Applying the Cauchy-Schwartz inequality on the right side, we get

$$\mu\|Du_\lambda-Du\|_{L^2(A)}\le\Big[||f||_{L^\infty(\Omega)}+g_0C_P(\frac{1}{\lambda^n}-\frac{1}{\lambda})\Big]|A|^{1/2}.\qquad(2.9)$$

We recall the following lemma of Sobolev: For $1 \leq \alpha < n$, there is a constant S_α depending only upon α, n and Ω such that, for every $w \in W_0^{1,\alpha}(\Omega)$, we have

$$\|w\|_{L^{\alpha^*}(\Omega)} \leq S_\alpha \|Dw\|_{L^\alpha(\Omega)},$$

where α^* denotes the Sobolev conjugate defined by $1/\alpha^* = 1/\alpha - 1/n$.

We now observe that

$$\|u_\lambda - u\|_{L^1(A)} \leq \|u_\lambda - u\|_{L^{1^*}(A)} |A|^{1-1/1^*}$$

(by Hölder's inequality)

$$\leq S_1 \|Du_\lambda - Du\|_{L^1(A)} |A|^{1/n}$$

(by Sobolev's Lemma, with $w = (u_\lambda - u)^+$)

$$\leq S_1 \|Du_\lambda - Du\|_{L^2(A)} |A|^{1/2+1/n},$$

by Hölder's inequality. Then, using this in (2.9), we get

$$\|u_\lambda - u\|_{L^1(A)} \leq C\Big[||f||_{L^\infty(\Omega)} + (\frac{1}{\lambda^n} - \frac{1}{\lambda})\Big] |A|^\gamma,$$

with $\gamma := 1 + 1/n > 1$ and for some constant C which depends only on Ω, g_0, and μ. By Lemma 2 we have $||f||_{L^\infty(\Omega)} \leq C_0(1-\lambda)$. Moreover, $(1/\lambda^n - 1/\lambda)$ is bounded above by $C_1(1-\lambda)$ (where C_1 depends only on n (recall that $\lambda \geq 1/2$). Thus

$$\|u_\lambda - u\|_{L^1(A)} \leq C_2(1-\lambda)|A|^\gamma,$$

with $C_2 := C(C_0 + C_1)$.

Now let us denote A by $A(q)$ to display its dependence on q. Put $\rho(q) := |A(q)|$. Then ρ is a nonnegative, nonincreasing function such that $\rho(q) \to 0$ when $q \to +\infty$. Moreover, we have for any $q > \bar{q}$, thanks to Fubini's theorem,

$$\int_q^{+\infty} \rho(t)\, dt = \frac{1}{1-\lambda} \int_{A(q)} |u_\lambda - u|\, dy \leq C_2 \rho(q)^\gamma. \tag{2.10}$$

We now require the following result (cf. Hartman and Stampacchia [47]):

Lemma 3 Let ρ be a nonnegative, nonincreasing function on $[0, +\infty)$ such that $\rho(t) \to 0$ as $t \to +\infty$ and

$$\int_q^{+\infty} \rho(t)\, dt \leq c\rho(q)^\gamma,\ q > \bar{q},$$

where $c > 0, \gamma > 1$ are constants. Then $\rho(t) = 0$ for

$$t > c\gamma\rho(\bar{q})^{\gamma-1}/(\gamma - 1) + \bar{q}.$$

To see this, note that the function $H(q) := \int_q^{+\infty} \rho(t)\,dt$ is absolutely continuous and satisfies

$$H'(q) = -\rho(q) \leq -[H(q)/c]^{1/\gamma}.$$

Then

$$G(q) := \gamma H^{(\gamma-1)/\gamma}(q)/(\gamma-1) + q/c^{1/\gamma}$$

has $G'(q) \leq 0$ for $q > \bar{q}$, as long as $H > 0$. For such q we may therefore write

$$0 \leq \gamma H^{1-1/\gamma}(q)/(\gamma-1) \leq \gamma H^{1-1/\gamma}(\bar{q})/(\gamma-1) - (q-\bar{q})/c^{1/\gamma}.$$

Consequently, $H(q) = 0$ for every

$$q \geq q_0 := \gamma c^{1/\gamma} H^{1-1/\gamma}(\bar{q})/(\gamma-1) + \bar{q},$$

in which case $\rho(t) = 0$ for $t > q_0$. The lemma follows from the fact that $H(\bar{q}) \leq c\rho(\bar{q})^\gamma$.

Applying this lemma to (2.10), we deduce that for any choice of q_0 satisfying

$$q_0 > C_2|\Omega|^{\gamma-1}\gamma/(\gamma-1) + \bar{q},$$

we have $|A(q)| = 0$ if $q \geq q_0$. We may summarize the current state of the proof as follows: for any choice of $z \in \Gamma$, we have, almost everywhere on $\Omega_\lambda := \lambda(\Omega - z) + z$, the inequality

$$u_\lambda(x) := \lambda u((x-z)/\lambda + z) - q_0(1-\lambda) \leq u(x).$$

Note that q_0 does not depend on λ, so that this assertion is true for any $\lambda \in [1/2, 1)$.

The final step in the proof is to deduce from this that u is locally Lipschitz in Ω. Let $x_0 \in \Omega$, and let $x, y \in B(x_0, d_\Gamma(x_0)/8)$ be two Lebesgue points for u; thus x, for example, satisfies

$$\lim_{\epsilon \to 0} \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} u(\omega)\,d\omega = u(x).$$

Let $z := \pi_\Gamma(y|x)$ be the unique point of Γ of the form $y + t(x-y)$ with $t \geq 0$. There exists $\lambda \in [1/2, 1)$ such that $y = (x-z)/\lambda + z$. Then $x \in \Omega_\lambda$. Let $\epsilon > 0$ such that $B(x,\epsilon) \subset \Omega_\lambda$. We have proved that for almost every $\omega \in B(x,\epsilon)$, we have

$$\lambda u((\omega - z)/\lambda + z) \leq u(\omega) + q_0(1-\lambda).$$

Integrating this relation over $B(x,\epsilon)$ and dividing by $|B(x,\epsilon)|$, we get

$$\frac{\lambda}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} u((\omega-z)/\lambda + z)\,d\omega \leq \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} u(\omega)\,d\omega + q_0(1-\lambda)$$

which, by a change of variables, is equivalent to

$$\frac{\lambda}{|B(y,\frac{\epsilon}{\lambda})|} \int_{B(y,\frac{\epsilon}{\lambda})} u(\omega)\,d\omega \leq \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} u(\omega)\,d\omega + q_0(1-\lambda).$$

When $\epsilon \to 0$, we get $\lambda u(y) \leq u(x) + q_0(1-\lambda)$, so that

$$u(y) \leq u(x) + Q\frac{|x-y|}{|y - \pi_\Gamma(y|x)|}, \tag{2.11}$$

with $Q := q_0 + ||u||_{L^\infty(\Omega)}$.

This inequality holds for almost all $x, y \in B(x_0, d_\Gamma(x_0)/8)$, since Lebesgue points for u constitute a set of full measure. It follows that u admits a locally Lipschitz representative for which (2.11) holds everywhere in Ω, and the theorem is proved.

Corollary The solution u satisfies

$$|Du(x)| \le \frac{Q}{d_\Gamma(x)},\ x \in \Omega \text{ a.e.}, \tag{2.12}$$

where Q depends on $\|u\|_{L^\infty(\Omega)}$ and the data of the problem (P).

2.3 A variant of the theorem

The hypothesis (HG) used in the proof of Theorem 2.1 included the differentiability of G with respect to u. A natural approach to removing that condition is to approximate G by a smooth function G_i via mollification, apply the theorem in the differentiable case to the solution u_i of the perturbed problem (P_i), and then to pass to the limit. However, this line of argument requires an existence theorem for the perturbed problem, and one must also verify that the resulting Lipschitz condition for its solution u_i depends in a suitably stable way upon the data.

As regards existence, the required elements are provided for the most part in the results of Stampacchia [84], which can be adapted for the purpose described above. Following [84] (but without assuming differentiability) we introduce the hypothesis

$(HG)'$ G is measurable and we have

$$G(x,v) \ge -q|v|^2 - Q(x)|v|^\delta - R(x),$$

where $R \in L^1(\Omega), \delta \in (0,2), Q \in L^t(\Omega)$, with $1/t = 1 - \delta/2 + \delta/n$ and $q < \Lambda\mu/2$, where

$$\Lambda := \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |Du|^2 dx}{\int_\Omega |u|^2 dx}.$$

Further, G is locally Lipschitz in u in the following sense: there exists $M > 0$ such that for any $u, u' \in \mathbb{R}$ and almost all $x \in \Omega$, one has

$$|G(x,u) - G(x,u')| \le M|u - u'|(1 + |u|^\beta + |u'|^\beta),$$

with $0 \le \beta < 2^* - 1$, where $1/2^* = 1/2 - 1/n$ if $n > 2$, and 2^* is any number greater than 2 if $n = 2$. Finally, we assume there is a function $\bar{u} \in W^{1,2}(\Omega)$ admissible for (P) such that $I(\bar{u}) < +\infty$.

We say that u solves (P) relative to $W^{1,2}(\Omega)$ if u is itself in that class, and if we have $I(u) \le I(w)$ for all $w \in \phi + W_0^{1,2}(\Omega)$.

Theorem 2.3
Under hypotheses $(H\Omega)$, (HF), and $(HG)'$, there exists a solution to problem (P) relative to $W^{1,2}(\Omega)$. Any such solution u is bounded, and is a solution of (P) relative to $L^\infty(\Omega)$; further, if ϕ satisfies the Lower Bounded Slope Condition, then u is locally Lipschitz in Ω.

The fact that a solution u_0 exists is Theorem 8.1 in [84]. As indicated above, the first step in the proof is to approximate G by a smooth function G_i; a term $|u - u_0(x)|^2$ is added to assure convergence of the solution u_i of the perturbed problem to u_0. The existence theorem in [84] must be detailed more completely in order to observe the stability of the estimates with respect to the type of perturbations present (in particular, the provenance of the bound on $\|u_i\|_{L^\infty(\Omega)}$ must be carefully traced). Then Theorem 2.1 is applied to deduce the Lipschitz condition (2.12), which carries over in the limit to u_0. We omit the essentially routine details of this proof (see the first appendix to Chapter 2).

2.4 Continuity at the boundary

The proof of Theorem 2.1 provided a Lipschitz constant for the solution u (see (2.12)) that goes to infinity at the boundary. We know by example that in general u fails to be globally Lipschitz, so this must be expected. But there remains the question of whether u is continuous at the boundary. Such a continuity conclusion cannot result from (2.12) alone, but it turns out that the directional nature of the Lipschitz condition (2.11), together with the barrier provided by Theorem 2.2, provides the extra information needed to obtain boundary continuity in a number of special cases. The arguments of [28] go through with no change, so we content ourselves here with recording the results. Note that the issue of continuity at the boundary does not arise in the classical setting with BSC, since then the solution is globally Lipschitz on Ω.

The theorems below introduce the hypothesis that u belongs to $W^{1,p}(\Omega)$. Under the hypotheses of either Theorem 2.1 or 2.3, this is easily seen to hold whenever F satisfies, for certain positive constants σ and N,

$$F(v) \geq \sigma |v|^p - N \quad \forall\, v \in \mathbb{R}^n.$$

Our hypothesis (HF) already guarantees that this holds for $p = 2$.

Theorem 2.4
In addition to the hypotheses of either Theorem 2.1 or 2.3, assume that Γ is a polyhedron. Then any solution u of (P) is Hölder continuous on $\overline{\Omega}$ of order $1/(n+2)$. If moreover $u \in W^{1,p}(\Omega)$ with $p > 2$, then u satisfies on $\overline{\Omega}$ a Hölder condition of order

$$a := \frac{p-1}{n+2p-2}.$$

Theorem 2.5
In addition to the hypotheses of either Theorem 2.1 or 2.3, assume that Γ is $C^{1,1}$ and that u is a solution of (P) lying in $W^{1,p}(\Omega)$, with $p > (n+1)/2$. Then u satisfies on $\overline{\Omega}$ a Hölder condition of order

$$b := \frac{2p-n-1}{4p+n-3}.$$

Under merely the hypotheses of Theorem 2.1 or 2.3, it is an open question whether a solution u of (P) must be continuous at the boundary.

Chapter 3

Local Lipschitz continuity of nonlinear differential-functional equations

This chapter is based on the paper *Local Lipschitz continuity of solutions of non-linear elliptic differential-functional equations* accepted at *ESAIM : COCV*.

3.1 Introduction

We study a Dirichlet boundary value problem associated with the following non-linear elliptic differential-functional equation :

$$\operatorname{div}[a(\nabla u)] + F[u] = 0. \tag{3.1}$$

We seek solutions in the space of functions $u \in W^{1,1}(\Omega)$ (where Ω is an open bounded convex set in $\mathbb{R}^n, n \geq 2$), whose trace $\operatorname{tr} u$ on $\Gamma := \partial\Omega$ is equal to some function $\phi : \Gamma \to \mathbb{R}$. For a fixed $x \in \Omega, F[u](x)$ is a non-linear functional of u. For example, Hartman and Stampacchia considers the Euler equation of the variational problem

$$\min\{\int_\Omega f(\nabla u)\, dx - \left[\int_\Omega h(x,u)\, dx\right]^\beta\}.$$

Then, $a_j(p) = f_{p_j}(p), F[u](x) = G[u]g(x,u)$ with

$$G[u] = \left[\int_\Omega h(x,u(x))\right]^{\beta-1} \quad \text{and} \quad g(x,u) = \beta h_u(x,u).$$

We say that $u \in W^{1,1}(\Omega)$ is a weak solution of (3.1) in $W^{1,1}_\phi(\Omega)$ (the space of functions in $W^{1,1}(\Omega)$ whose trace $\operatorname{tr} u$ is equal to the function ϕ) if $a(\nabla u) \in L^1_{\text{loc}}(\Omega), F[u] \in L^1_{\text{loc}}(\Omega)$ and

$$(E) \quad \int_\Omega \{\langle a(\nabla u(x)), \nabla\eta(x)\rangle - F[u](x)\eta(x)\}\, dx = 0$$

for all continuously differentiable η with compact support in Ω; that is, $\eta \in C_c^1(\Omega)$.

The problem (E) has been tackled by Hartman and Stampacchia, among many others, in [47], which will be a recurrent reference throughout this paper. There, the authors show the existence of solutions to (E) in the space $\mathrm{Lip}(\Omega, \phi)$ of uniformly Lipschitz continuous functions on $\bar{\Omega}$ whose trace on Γ is ϕ. Their proof is based on two main tools. The first one is an abstract existence theorem in functional analysis. This theorem enables them to assert for each $K > 0$, the existence of a solution u_K to (E) in the space $\mathrm{Lip}(\Omega, \phi, K)$ of uniformly Lipschitz continuous functions of Lipschitz rank no greater than K. The second tool of the proof is an a priori bound on the Lipschitz rank of u_K, independently of K. Then, Hartman and Stampacchia obtain the desired solution u in the space $\mathrm{Lip}(\Omega, \phi)$ as a limit, as $K \to \infty$, of the sequence (u_K).

We are mainly interested in the generalization of the second tool: the *a priori* bound on the Lipschitz rank. In [47], it is based on a maximum principle on the gradient of the solutions, which can be stated as follows (see Lemma 10.0 in [47]):

$$||\nabla u||_{L^\infty(\Omega)} \leq \sup_{x \in \Gamma, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|} + C, \tag{3.2}$$

where C is a constant depending on the data of the problem.

This maximum principle had already appeared in a variational context (see [69]) to give a proof of the Hilbert-Haar theorem. It is based on a device due to Rado which amounts to the comparison of a solution u and a *translated version* of u, say $u_\tau := u(\cdot + \tau)$ which is (nearly) a solution of the same equation but on $\Omega_\tau := \Omega - \tau$.

To estimate the right hand side of (3.2), Hartman and Stampacchia consider the barrier technique. This technique has been widely used in the theory of elliptic pde's (see [36]). In particular, Lieberman (see [54],[55], [56]) has studied the relationship between the regularity of ϕ on Γ and the regularity of the solutions on Ω. Nevertheless, he always posits assumptions on the *upper growth* of a, which is not the case in our main result.

In [47], different types of hypotheses on ϕ are considered. One of them requires that ϕ satisfy the *bounded slope condition* (BSC). The BSC of rank Q is the assumption that, given any point $\gamma \in \Gamma$, there exist two affine functions

$$y \mapsto \langle \zeta_\gamma^-, y - \gamma \rangle + \phi(\gamma), \;\; y \mapsto \langle \zeta_\gamma^+, y - \gamma \rangle + \phi(\gamma)$$

agreeing with ϕ at γ, whose slopes satisfy $|\zeta_\gamma^-| \leq Q, |\zeta_\gamma^+| \leq Q$, and such that

$$\langle \zeta_\gamma^-, \gamma' - \gamma \rangle + \phi(\gamma) \leq \phi(\gamma') \leq \langle \zeta_\gamma^+, \gamma' - \gamma \rangle + \phi(\gamma), \quad \forall \gamma' \in \Gamma.$$

This condition forces ϕ to be affine on 'flat parts' of Γ. Moreover, if Ω is smooth, then it forces ϕ to be smooth as well (see Hartman [43] for precise statements; see also [12]).

Recently, Clarke [28] has introduced a new hypothesis on ϕ, the *lower bounded slope condition* (LBSC) of rank Q : given any point $\gamma \in \Gamma$, there exists an affine function

$$y \mapsto \langle \zeta_\gamma, y - \gamma \rangle + \phi(\gamma),$$

with $|\zeta_\gamma| \leq Q$ such that

$$\langle \zeta_\gamma, \gamma' - \gamma \rangle + \phi(\gamma) \leq \phi(\gamma'), \quad \forall \gamma' \in \Gamma.$$

This requirement enlarges the class of boundary functions which it allows, compared to the BSC. It can be shown in particular that $\phi : \Gamma \to \mathbb{R}$ satisfies the LBSC if and only if it is the restriction to Γ of a convex function. When Ω is uniformly convex, ϕ satisfies the LBSC if and only if it is the restriction to Γ of a semiconvex function (see [12] for details and further properties).

Clarke has shown in a variational context that the LBSC gives the local Lipschitz continuity of minimizers (see [28], see also [13]). The proof rests on a modification of Rado's device: The minimizer u is compared now to a *dilated* version of u (and not to a translated one).

The goal of this paper is to adapt the ideas appearing in [28] and [13], used in a variational context, to our present setting, so as to prove existence and local Lipschitz regularity of the solutions to the elliptic differential-functional equations considered above, when the LBSC is satisfied (rather than the BSC). We remark that local Lipschitzness is the crucial property to show further regularity results with the help of the De Giorgi's theory, when the data are regular enough (see [47], section 14). In our context, however, we can only get *local* regularity; that is, on any compact subsets of Ω.

The next section describes the hypotheses that we posit on the data, and the proof of our theorem is given in section 3. The final section discusses the issue of the continuity of the solution at the boundary.

3.2 The main result

Recall that $\mathrm{Lip}(\Omega)$ denotes the set of uniformly Lipschitz continuous functions on Ω (or, equivalently, on $\bar{\Omega}$). Let $\mathrm{Lip}(\Omega, \phi)$ be the set of functions $u \in \mathrm{Lip}(\Omega)$ for which $u = \phi$ on Γ. For a given K, let $\mathrm{Lip}(\Omega, \phi, K)$ be the set of functions $u \in \mathrm{Lip}(\Omega, \phi)$ of rank $\leq K$ (this set being empty if ϕ is not Lipschitz of rank at most K). We now specify the hypotheses on the data of the problem (E). Recall that

$(H\Omega)$ Ω is an open bounded convex set in $\mathbb{R}^n, n \geq 2$.

$(H\phi)$ ϕ satisfies the lower bounded slope condition of rank Q.

This implies that ϕ can be extended as a convex function on $\mathbb{R}^n$, which will be done henceforth. Moreover, we may assume that ϕ is globally Lipschitz of rank Q. As Ω is convex, it has a Lipschitz boundary, which justifies the use of trace in the boundary condition: $\mathrm{tr}\, u = \phi$.

We will assume that $a = (a_1, .., a_n)$ is continuous on $\mathbb{R}^n$ and satisfies

$$(Ha) \qquad \langle a(p) - a(q), p - q \rangle \geq \mu_0 |p - q|^2,$$

for some $\mu_0 > 0$. This implies (with $q = 0$) that for any $\epsilon > 0$, there exists $N_\epsilon \geq 0$ such that $\langle a(p), p \rangle \geq (\mu_0 - \epsilon)|p|^2 - N_\epsilon$.

The non-linear functional F satisfies the four hypotheses below (where u is any bounded and continuous function on Ω):

$(HF0)$ $x \in \Omega \mapsto F[u](x)$ is well-defined and measurable,

$$(HF1) \quad F[u](x) \operatorname{sgn} u(x) \leq \sum_{i=1}^{m} c_i ||u||^{\beta(i)}_{L^{\alpha(i)}(\Omega)} |u(x)|^{\gamma(i)-1}, \; x \in \Omega \text{ a.e.}$$

where $c_i \geq 0, \alpha(i) \geq 1, \beta(i) \geq 0, \gamma(i) \geq 1$ and $\alpha(i) \leq 2^*, \beta(i) + \gamma(i) \leq 2$. (As usual, $1/2^* = 1/2 - 1/n$ when $n > 2$. If $n = 2, 2^*$ denotes any number larger than 4). We also assume that the coefficients c_i in $(HF1)$ satisfy

$$\mu_0 - \sum{}' c_i \Lambda^{-2} |\Omega|^{1-2/\sigma + \beta(i)/\alpha(i)} > 0 \tag{3.3}$$

where $\sum'$ is the sum over the indices i for which $\beta(i) + \gamma(i) = 2$. Here,

$$\sigma := \max_{i=1,..,m} (\alpha(i), 2) \leq 2^*$$

and

$$\Lambda := \inf_{u \in W_0^{1,2}(\Omega)} \frac{||\nabla u||_{L^2(\Omega)}}{||u||_{L^\sigma(\Omega)}}.$$

Furthermore, we assume that for every number $M > 0$, there exists a number $\chi(M)$ such that

$$(HF2) \qquad |u(x)| \leq M \ \text{ on } \ \Omega \Rightarrow |F[u](x)| \leq \chi(M).$$

The last hypothesis on F is:

$(HF3)$ If $u_h \in \mathrm{Lip}(\Omega, \phi)$ for $h = 1, 2, ...$ is a bounded sequence in $L^\infty(\Omega)$ which converges to u uniformly on compact subsets of Ω as $h \to \infty$, then $F[u_h](x) \to F[u](x)$ a.e. on Ω.

These hypotheses are closely related to those of Hartman and Stampacchia [47]. They are satisfied by the example given in the introduction.

We can pick some $\epsilon > 0$ such that inequality (3.3) remains true when μ_0 is replaced by $\mu := \mu_0 - \epsilon > 0$. With that μ, (Ha) remains true and we have

$$\langle a(p), p \rangle \geq \mu |p|^2 - N \tag{3.4}$$

for some $N > 0$.

Under these hypotheses, we can state our theorem :

Theorem 3.1 *Under hypotheses* $(H\Omega), (H\phi), (Ha)$ *and* $(HF0), (HF1), (HF2), (HF3)$, *there exists a locally Lipschitz* $u \in W_\phi^{1,2}(\Omega) \cap L^\infty(\Omega)$ *which satisfies* (E) :

$$\int_\Omega \{\langle a(\nabla u(x)), \nabla \eta(x) \rangle - F[u](x)\eta(x)\}\, dx = 0 \quad \forall \eta \in C_c^1(\Omega).$$

This theorem generalises Theorem 12.1 in the article of Hartman and Stampacchia [47], in the sense that the bounded slope condition is reduced to the lower bounded slope condition. In contrast to [47],however, we do not assert the global Lipschitzness of the solution. This explains why the hypotheses that we have made on a and F are more restrictive than those appearing in [47]. In particular, a small dependence on the gradient is allowed there in the hypothesis corresponding to $(HF2)$.

In fact, it is not the case in our context that solutions are globally Lipschitz, as evidenced by the following example (see [12],[28]):

Example 3.1 *The set* Ω *is the open disc in* $\mathbb{R}^2$, $\phi(\cos\theta, \sin\theta) := -\pi^2/6 + \pi/2\theta - \theta^2/4, 0 \leq \theta < 2\pi$, $F = 0$ *and* $a(p) = p$. *Then the solution of* (E) *is locally Lipschitz but not globally Lipschitz.*

Remark 3.1 *There is another version of the theorem where the lower bounded slope condition is replaced by an upper bounded slope condition.*

3.3 Proof of the theorem

Following the terminology of [47], by a K quasi solution of (3.1) will be meant a function $u \in \mathrm{Lip}(\Omega, \phi, K)$ satisfying

$$\int_{\Omega} \{\langle a(\nabla u), \nabla(v-u)\rangle - F[u](v-u)\} \geq 0, \quad \forall v \in \mathrm{Lip}(\Omega, \phi, K). \tag{3.5}$$

We recall here some results of [47]. First, the following existence theorem holds (this is [47], Lemma 12.1.).

Proposition 3.1 *For every $K > Q$, there exists a K quasi solution to (3.5).*

The following proposition (which is exactly Theorem 8.1 in [47]) provides an a priori bound in $L^\infty(\Omega)$ for any K quasi solution ($K > Q$).

Proposition 3.2 *There exists a constant T (independent of K) such that if u is a K quasi solution of (3.5), then*

$$|u(x)| \leq T, \quad \textit{on} \;\; \Omega.$$

From this bound, we can infer easily an *a priori* bound in $W^{1,2}(\Omega)$:

Proposition 3.3 *There exists a constant T' (independent of $K > Q$) such that if u is a K quasi solution of (3.5), then*

$$||u||_{W^{1,2}} \leq T'.$$

Proof: Since $\phi \in \mathrm{Lip}(\Omega, \phi, K)$, we have

$$\int_{\Omega} \langle a(\nabla u_K), \nabla(u_K - \phi)\rangle \leq \int_{\Omega} F[u_K](u_K - \phi)$$

so that

$$\int_{\Omega} \langle a(\nabla \phi), \nabla(u_K - \phi)\rangle + \mu \int_{\Omega} |\nabla(u_K - \phi)|^2 \leq \int_{\Omega} F[u_K](u_K - \phi).$$

Then (using the fact that $||u_K - \phi||_{L^\infty(\Omega)} \leq T + ||\phi||_{L^\infty(\Omega)}$),

$$\mu \int_{\Omega} |\nabla(u_K - \phi)|^2 \leq ||a(\nabla\phi)||_{L^\infty(\Omega)} \int_{\Omega} |\nabla(u_K - \phi)| + \chi(T)(T + ||\phi||_{L^\infty(\Omega)})$$

where $\chi(T)$ is given by $(HF2)$. Writing that

$$||a(\nabla\phi)||_{L^\infty(\Omega)} \int_{\Omega} |\nabla(u_K - \phi)| \leq$$

$$\epsilon \int_{\Omega} |\nabla(u_K - \phi)|^2 + ||a(\nabla\phi)||^2_{L^\infty(\Omega)} |\Omega|/(4\epsilon),$$

we see that $||\nabla(u_K - \phi)||_{L^2(\Omega)}$ is bounded by a constant which depends on $||a(\nabla\phi)||_{L^\infty(\Omega)}$, μ, T, $||\phi||_{L^\infty(\Omega)}$ and Ω. Hence, (u_K) is bounded in $W^{1,2}(\Omega)$. This completes the proof.

□

The proof of Theorem 3.1 uses the well-known barrier technique:

Proposition 3.4 *There exists $\bar{Q} \geq 0$ such that for any $\gamma \in \Gamma$, there exists $w : \bar{\Omega} \to \mathbb{R}$ Lipschitz of rank $\bar{Q}$ which satisfies*

$$w(\gamma) = \phi(\gamma), w(y) \leq u(y) \quad \forall y \in \Omega,$$

for any K quasi solution u of (3.5) and any $K > \bar{Q}$.

Proof : We build the same barrier as in [13], Theorem 2.2. There is an element ζ with $|\zeta| \leq Q$ in the subdifferential of ϕ at γ :

$$\phi(x) - \phi(\gamma) \geq \langle \zeta, x - \gamma \rangle \quad \forall x \in \mathbb{R}^n.$$

By $(HF2)$ and Proposition 3.2, $|F[u](x)| \leq \chi(T)$ $x \in \Omega$ a.e., for any K quasi-solution u of (3.5). Fix any $R > (\chi(T) + 1) \exp(\operatorname{diam} \Omega)/\mu$ where μ is given by (3.4). Recall that (Ha) remains true when μ_0 is replaced by μ. Let ν be a unit outward normal vector to $\bar{\Omega}$ at γ and define

$$w(x) := \phi(\gamma) + \langle \zeta, x - \gamma \rangle - R\{1 - \exp(\langle x - \gamma, \nu \rangle)\}.$$

The function w agrees with ϕ at γ and is Lipschitz of rank

$$\bar{Q} := Q + R \exp(\operatorname{diam} \Omega).$$

Let $K > \bar{Q}$ and u be a K quasi solution of (3.5). We have to show that the set

$$S := \{x \in \Omega : w(x) > u(x)\}$$

has measure 0. The function $M(x) := \max[u(x), w(x)]$ is Lipschitz of rank K and its trace is ϕ (this follows from the subgradient inequality for ζ and the fact that $\langle x - \gamma, \nu \rangle \leq 0$ for $x \in \Omega$).

As u is a K quasi solution of (3.5) (relative to M), we have

$$\int_S \langle a(\nabla u), \nabla(u - w) \rangle \leq \int_S F[u](u - w). \tag{3.6}$$

Thanks to (Ha), we get

$$\int_S \langle a(\nabla w), \nabla(u - w) \rangle \leq \chi(T) \int_S (w - u) \tag{3.7}$$

Let us make the temporary assumption that a is C^1. Then, a straightforward calculation yields

$$\begin{aligned} \operatorname{div}[a(\nabla w)] &= R \exp(\langle x - \gamma, \nu \rangle) \sum_{i,j} \partial_{p_j} a_i(\nabla w) \nu_i \nu_j \\ &\geq \chi(T) + 1, \end{aligned}$$

in light of (Ha) and because of how R was chosen. Then, (3.7) implies:

$$\begin{aligned} \chi(T) \int_S (w - u) &\geq \int_S \langle a(\nabla w), \nabla(u - w) \rangle \\ &\geq \int_S (w - u) \operatorname{div}[a(\nabla w)] \\ &\geq (\chi(T) + 1) \int_S (w - u). \end{aligned}$$

This shows that S is of measure 0, since $w-u>0$ on S.

In the general case in which a is not C^1, we consider a sequence a_k of C^1 vector fields converging to a uniformly on compact sets and satisfying (Ha). Then, for each k,

$$\int_S \langle a_k(\nabla w), \nabla(u-w)\rangle \geq (\chi(T)+1)\int_S (w-u)$$

and the quantity $\int_S \langle a_k(\nabla w), \nabla(u-w)\rangle$ converges to $\int_S \langle a(\nabla w), \nabla(u-w)\rangle$ as k goes to $+\infty$. This shows that the result is still true when a is merely assumed continuous.

□

We then proceed exactly as in [13]. Consider a K quasi solution u of (3.5). Let $\lambda \in [1/2,1), q > \bar{q} := \bar{Q}\mathrm{diam}\,\Omega + ||\phi||_{L^\infty(\Omega)}$ and $z \in \Gamma$. We will denote by

$$u_\lambda(x) \quad := \quad \lambda u((x-z)/\lambda + z) - q(1-\lambda) \tag{3.8}$$

$$\Omega_\lambda \quad := \quad \lambda(\Omega - z) + z. \tag{3.9}$$

Then u_λ belongs to the space $\mathrm{Lip}(\Omega_\lambda, \phi_\lambda, K)$, where $\phi_\lambda(x) := \lambda\phi((x-z)/\lambda + z) - q(1-\lambda)$. We want to compare u_λ and u on $\Gamma_\lambda := \partial\Omega_\lambda$.

This is done by the following proposition, whose proof appears in [13]:

Proposition 3.5 *We have $u_\lambda \leq u$ on Γ_λ.*

The next step of the proof is to show that the set

$$A := \{y \in \Omega_\lambda : u_\lambda(y) > u(y)\}$$

has measure zero. Once again, the proof is very similar to that of [13]:

By definition of a K quasi solution, $u \in \mathrm{Lip}(\Omega,\phi,K)$ satisfies (3.5); that is,

$$\int_\Omega \langle a(\nabla u), \nabla(v-u)\rangle - F[u](v-u) \geq 0, \quad \forall v \in \mathrm{Lip}(\Omega,\phi,K).$$

We will denote $F[u](x)$ by $g(x)$ for all $x \in \Omega$. Then, $g \in L^\infty(\Omega)$.

Let $w(x) := \min(u_{|\Omega_\lambda}, u_\lambda) \in \mathrm{Lip}(\Omega_\lambda, \phi_\lambda, K)$ (thanks to Proposition 3.5).

Let $w^\lambda(x) := 1/\lambda w(\lambda(x-z)+z) + q(1/\lambda - 1) \in \mathrm{Lip}(\Omega,\phi,K)$. With $v := w^\lambda$ in (3.5), we get after an obvious change of variables

$$0 \leq \int_{\Omega_\lambda} \langle a(\nabla u_\lambda(y)), \nabla w(y) - \nabla u_\lambda(y)\rangle - g(\frac{y-z}{\lambda}+z)(\frac{w}{\lambda}(y) - \frac{u_\lambda}{\lambda}(y))\,dy$$

which implies

$$0 \leq \int_A \langle a(\nabla u_\lambda(y)), \nabla u(y) - \nabla u_\lambda(y)\rangle - \frac{1}{\lambda} g(\frac{y-z}{\lambda}+z)(u(y)-u_\lambda(y))\,dy.$$

Let $W(x) := \max(u_{|\Omega_\lambda}(x), u_\lambda(x))$ for $x \in \Omega_\lambda$ and $W(x) := u(x)$ for $x \in \Omega - \Omega_\lambda$. Then $W \in \mathrm{Lip}(\Omega,\phi,K)$. With $v := W$ in (3.5), we get

$$0 \leq \int_A \langle a(\nabla u), \nabla(u_\lambda - u)\rangle - g(y)(u_\lambda - u).$$

Summing these two last inequalities, we get

$$0 \leq \int_A \langle -a(\nabla u_\lambda) + a(\nabla u), \nabla(u_\lambda - u)\rangle + (\frac{1}{\lambda} g(\frac{y-z}{\lambda} + z) - g(y))(u_\lambda - u)$$

so that using (Ha),

$$\mu \int_A |\nabla(u_\lambda - u)|^2 \leq \int_A (\frac{1}{\lambda} g(\frac{y-z}{\lambda} + z) - g(y))(u_\lambda - u).$$

We proceed with the following lemma (the proof of which can be found in [13]; see the calculations following inequality (6) there):

Lemma 3.1 *Let $u \in W^{1,2}(\Omega), g \in L^\infty(\Omega)$ and $\mu > 0$. Assume that there exists $\bar{q}$ such that for any $q > \bar{q}$ and $\lambda \in [1/2, 1)$, we have*

$$\mu \int_A |\nabla(u_\lambda - u)|^2 \leq \int_A (\frac{1}{\lambda} g(\frac{y-z}{\lambda} + z) - g(y))(u_\lambda - u),$$

where $u_\lambda, \Omega_\lambda$ are defined as in (3.8),(3.9) and $A = A(q) := \{y \in \Omega_\lambda : u_\lambda(y) > u(y)\}$. Then, there exists $q_0 > \bar{q}$ such that $A(q)$ has measure 0 for any $q \geq q_0$. The number q_0 only depends on $\bar{q}, n, \mu, \Omega, ||g||_{L^\infty(\Omega)}$ but not on $u, \lambda \in [1/2, 1)$ nor on $z \in \Gamma$.

We infer from this lemma that there exists $q_0 > 0$ such that

$$\lambda u((x-z)/\lambda + z) \leq u(x) + q(1-\lambda) \quad \forall q \geq q_0. \tag{3.10}$$

This implies that the Lipschitz rank of u can be bounded independently of K on any compact set of Ω, as shown by the following lemma (for a proof of this one, see the final step of the proof of the main theorem in [13]):

Lemma 3.2 *Let $u \in L^\infty(\Omega)$. Assume that there exists $q_0 > 0$ such that for any $\lambda \in [1/2, 1), z \in \Gamma$, we have:*

$$\lambda u((y-z)/\lambda + z) - q_0(1-\lambda) \leq u(y),$$

a.e. $y \in \Omega_\lambda := \lambda(\Omega - z) + z$. Then, u (admits a representative which) is locally Lipschitz on Ω and we have

$$|Du(x)| \leq \frac{||u||_{L^\infty(\Omega)} + q_0}{d_\Gamma(x)}, \quad x \in \Omega \quad a.e.,$$

where d_Γ denotes the distance to Γ.

We may summarize the current state of the proof as follows: for each $K > 0$, there exists $u_K \in \mathrm{Lip}(\Omega, \phi, K)$ a K quasi solution of (3.5), such that $||u_K||_{L^\infty(\Omega)} \leq T$, $||u_K||_{W^{1,2}}(\Omega) \leq T'$ and the Lipschitz rank of u_K on any compact subset $\Omega_0 \subset \Omega$ is bounded by

$$\frac{T + q_0}{\mathrm{d}\,(\Omega_0, \Gamma)}$$

where q_0, T are independent of K and $\mathrm{d}\,(\Omega_0, \Gamma)$ denotes the distance between Ω_0 and Γ. Let Ω_j be an increasing sequence of open subsets of Ω satisfying

$\Omega_j \subset \bar{\Omega}_j \subset \Omega_{j+1}$ and $\cup_{j\geq 1}\Omega_j = \Omega$. Let K_j be a common Lipschitz rank for all the functions u_K restricted to Ω_j. Then, up to a subsequence, the functions u_K converge uniformly on every compact subset of Ω to a function u which is Lipschitz of rank K_j on Ω_j. Moreover, we can suppose that for every $j, \nabla u_K$ converges to ∇u in $\sigma(L^\infty(\Omega_j), L^1(\Omega_j))$. Finally, we can also assume that (u_K) converges weakly to u in $W^{1,2}(\Omega)$. It remains to show that:

Proposition 3.6 *The function u is a weak solution of (E) and $u \in W_0^{1,2}(\Omega)+\phi$.*

Proof : This second assertion is trivial in view of the weak convergence in $W^{1,2}(\Omega)$. Fix some $\eta \in C_c^\infty(\Omega)$. Let L be a Lipschitz constant for η. Let $\epsilon > 0$. We know that $||F[u_K]||_{L^\infty} \leq \chi(T)$. Let j be big enough so that $\operatorname{supp}\eta \subset \Omega_j$. Let ϵ_j be such that

$$\epsilon_j(L + 3K_{j+1} + 1)\sup_{|p|\leq L+3K_{j+1}+1}|a(p)| \leq \epsilon$$

and such that

$$\epsilon_j(\chi(T)(||\eta||_{L^\infty(\Omega)} + 2T)) < \epsilon.$$

Let Ω'_j be an open subset of Ω such that

$$\bar{\Omega}_j \subset \Omega'_j \subset \bar{\Omega}'_j \subset \Omega_{j+1}$$

and $|\Omega'_j/\Omega_j| < \epsilon_j$.

Let $\theta_j \in C_c^\infty(\Omega'_j)$ such that $\theta_j \equiv 1$ on Ω_j and $0 \leq \theta_j \leq 1$.

Let

$$\psi_K(x) := u_K(x) + \eta(x) + \theta_j(x)(u(x) - u_K(x)).$$

For any $K \geq 0, \psi_K(x) = u_K(x)$ on Ω/Ω'_j and $\psi_K(x) = \eta(x) + u(x)$ on Ω_j. A Lipschitz rank for ψ_K restricted to Ω'_j is $3K_{j+1} + L + |\nabla\theta_j|||u - u_K||_{L^\infty(\Omega_{j+1})} \leq L + 3K_{j+1} + 1$ for K sufficently large, say $K \geq M_j$ for some $M_j > 0$. Then, for any $K \geq \max(M_j, L + 3K_{j+1} + 1), \psi_K$ is K Lipschitz on Ω and we have:

$$\int_\Omega \langle a(\nabla u_K), \nabla(\psi_K - u_K)\rangle \geq \int_\Omega F[u_K](x)(\psi_K - u_K)(x)\,dx$$

which implies

$$\int_{\Omega'_j} \langle a(\nabla\psi_K), \nabla(\psi_K - u_K)\rangle \geq \int_{\Omega'_j} F[u_K](x)(\psi_K - u_K)(x)\,dx.$$

Hence,

$$\int_{\Omega_j} \langle a(\nabla(\eta + u)), \nabla(\eta + u - u_K)\rangle \geq \int_{\Omega_j} F[u_K](x)(\eta + \theta_j(u - u_K))(x)\,dx - 2\epsilon.$$

Passing to the limit when $K \to +\infty$ yields

$$\int_{\Omega_j} \langle a(\nabla(\eta + u)), \nabla\eta\rangle \geq \int_{\Omega_j} F[u](x)\eta(x)\,dx - 2\epsilon$$

(recall that $\nabla u \to \nabla u_K$ in $\sigma(L^\infty, L^1)$ and $F[u_K](x) \to F[u](x)$ a.e. and is bounded independently of K). In the previous inequality, we can replace Ω_j by Ω (since $\operatorname{supp}\eta \subset \Omega_j$) and notice that ϵ is arbitrary. We have then shown that

$$\int_\Omega \langle a(\nabla(\eta + u)), \nabla\eta\rangle \geq \int_\Omega F[u](x)\eta(x)\,dx.$$

Replace now η by $t\eta$ for any $t \in \mathbb{R} \setminus \{0\}$, divide by t and let $t \to 0$. Then

$$\int_\Omega \langle a(\nabla u), \nabla \eta \rangle = \int_\Omega F[u](x)\eta(x)\, dx.$$

This shows that u is a weak solution of (E) and completes the proof of the theorem.

3.4 Continuity at the boundary.

We know by example that in general, u fails to be globally Lipschitz. But there remains the question of whether u is continuous at the boundary. Under merely the hypotheses of Theorem 3.1, it is an open problem. However, the continuity on $\bar{\Omega}$ can be proved under additional hypotheses on Ω and/or on the integrability of u.

This conclusion is based on the following properties of u. First, there exists a function $\bar{w} \in \mathrm{Lip}\,(\Omega, \phi, \bar{Q})$ (for some $\bar{Q} > 0$) such that $w \leq u$ on Ω. Indeed, if we denote by w_γ the function built in Proposition 3.4, then the function $\bar{w} := \inf\limits_{\gamma \in \Gamma} w_\gamma$ belongs to $\mathrm{Lip}\,(\Omega, \phi, \bar{Q})$ and satisfies:

$$w_\gamma \leq u_K \quad , \quad \forall K > \bar{Q}.$$

So, the same inequality is satisfied by u instead of u_K.

Secondly, inequality (3.10) easily implies

$$u(y) \leq u(x) + (q_0 + ||u||_{L^\infty(\Omega)})\frac{|x-y|}{|y-z|}$$

whenever $|y - x| < 1/2|y - z|$, with $z \in \Gamma$ such that $y = (x - z)/\lambda + z$ for some $\lambda \in (1/2, 1)$.

The arguments of [28] (namely, the proofs of Theorem 2.2, 2.3 and 2.4) show the following:

Theorem 3.2 *If Γ is a polyhedron, then u is Hölder continuous on $\bar{\Omega}$ of order $1/(n+2)$. If Γ is $C^{1,1}$ and $u \in W^{1,p}$ with $p > (n+1)/2$, then u satisfies on $\bar{\Omega}$ a Hölder condition of order*

$$a := \frac{2p - n - 1}{4p + n - 3}.$$

Chapter 4

Fractional Sobolev spaces and Topology

This chapter is based on the paper *Fractional Sobolev Spaces and Topology* (accepted for publication in Nonlinear Analysis : TMA).

4.1 Introduction

Let M and N be compact connected smooth boundaryless Riemannian manifolds. Throughout the paper we assume that $m := \dim M \geq 2$. Our functional framework is the Sobolev space $W^{s,p}(M,N)$ which is defined by considering N as smoothly embedded in some Euclidean space $\mathbb{R}^l$ and then

$$W^{s,p}(M,N) = \{u \in W^{s,p}(M,\mathbb{R}^l) : u(x) \in N \text{ a.e.}\},$$

with $1 \leq p < \infty, 0 < s$. The space $W^{s,p}(M,N)$ is equipped with the standard metric $\mathrm{d}\,(u,v) = ||u-v||_{W^{s,p}}$. The main purpose of this paper is to determine whether or not $W^{s,p}(M,N)$ is path-connected and if not, when two elements u and v in $W^{s,p}(M,N)$ can be continuously connected in $W^{s,p}(M,N)$; that is, when there exists $H \in C^0([0,1], W^{s,p}(M,N))$ such that $H(0) = u$ and $H(1) = v$. If this is the case, we say that 'u and v are $W^{s,p}$ connected' (or $W^{s,p}$ homotopic).

Homotopy theory in the framework of Sobolev spaces is essential when studying certain problems in the calculus of variations. This is the case when the admissible functions are defined on a manifold M into a manifold N. One may hope to find multiple minimizers to these problems, ideally one in each homotopy class (see [88], [89] and also [17]).

The topology of $W^{s,p}(M,N)$ depends on two features of the problem, namely the topology of M and N, and the value of s and p. When $s = 1$, the study of the topology of $W^{1,p}(M,N)$ was initiated in [20] . The analysis of homotopy classes (for $s = 1$) was subsequently tackled in [40] (see also [88], [89] for related and earlier results). These results have been generalized to $W^{s,p}(M,N)$ for non integer values of s and $1 < p < \infty$ when M is a smooth, bounded, connected open set in an Euclidean space and when $N = S^1$ (see [14]). In this case, the proofs exploit in an essential way the fact that the target manifold is S^1. In contrast, our main concern is to determine to what extent the methods of [40] and the tools of [20] can be adapted to the case $s \neq 1$. Throughout the paper, we assume that $0 < s < 1 + 1/p$ or $sp \geq \dim M$.

Our first result gives some conditions which imply that $W^{s,p}(M, N)$ is path-connected:

Theorem 4.1 *Let $0 < s < 1 + 1/p$. Then the space $W^{s,p}(M, N)$ is path-connected when $sp < 2$.*

When $s = 1$, this result was proved in [20], where the condition $p < 2$ (for $s = 1$) is seen to be sharp. For instance, $W^{1,2}(S^1 \times \Lambda, S^1)$, where Λ is any open connected set, is not path connnected.

In the case $sp \geq 2$, we have:

Theorem 4.2 *Assume that $0 < s < 1 + 1/p, 2 \leq sp < \dim M$ and that there exists $k \in \mathbb{N}$ with $k \leq [sp] - 1$ such that $\pi_i(M) = 0$ for $1 \leq i \leq k, \pi_i(N) = 0$ for $k + 1 \leq i \leq [sp] - 1$. Then the space $W^{s,p}(M, N)$ is path-connected.*

The case $s = 1$ of the above theorem is Corollary 1.1 in [40].

More generally, it is natural to compare the connected components of the space $W^{s,p}(M, N)$ to those of $C^0(M, N)$. In certain cases, this is indeed possible:

Theorem 4.3 *a) If $sp \geq \dim M$ then $W^{s,p}(M, N)$ is path connected if and only if $C^0(M, N)$ is path connected.*
b) The $W^{s,p}$ homotopy classes are in bijection with the C^0 homotopy classes when $0 < s < 1 + 1/p, 2 \leq sp < \dim M$ and $\pi_i(N) = 0$ for $[sp] \leq i \leq \dim M$.

The statement a) is well-known and can be proved as in the appendix of [20]. Part b) for $s = 1$ was obtained in [40], Corollary 5.2.

When $s = 1$, Theorem 4.2 and Theorem 4.3 are particular cases of a more general result in [40] which asserts that there is a one-to-one map from the connected components of $W^{1,p}(M, N)$ into the connected components of the space $C^0(M^{[p]-1}, N)$. Here, $M^{[p]-1}$ denotes a $[p]-1$ skeleton of M. This may be re-expressed as follows: two maps u and v in $W^{1,p}(M, N)$ are $W^{1,p}$ homotopic if and only if u is $[p] - 1$ homotopic to v. For an accurate definition of $[p] - 1$ homotopy, one should refer to [40] or to section 4.6. Roughly speaking, this means that for a generic $[p] - 1$ skeleton $M^{[p]-1}$ of $M, u|_{M^{[p]-1}}$ and $v|_{M^{[p]-1}}$ are homotopic. This makes sense because for a generic $[p] - 1$ skeleton, u and v are both $W^{1,p}$ on these skeletons and hence continuous, by the Sobolev embedding. There is a corresponding version of this result in which $W^{1,p}$ is replaced by $W^{s,p}$:

Theorem 4.4 *Assume that $0 < s < 1 + 1/p, 2 \leq sp < \dim M$. Let $u, v \in W^{s,p}(M, N)$. Then u and v are $W^{s,p}$ connected if and only if u is $[sp] - 1$ homotopic to v.*

The techniques in [40] can be adapted in order to prove not only Theorem 4.4 but also the more general result where the condition $2 \leq sp < \dim M$ is replaced by: $0 < sp < \dim M$, and $sp \neq 1$. In turn, this last result implies Theorem 4.1 when $sp < 2, sp \neq 1$. However, the case $sp = 1$ seems delicate to handle via these techniques. This is the reason why we give a proof of Theorem

4.1 based on the tools of [20]. Besides its independent interest, it turns out that the technical core of the proof of Theorem 4.1 is also the technical core of the proof of Theorem 4.4. Furthermore, the techniques in [20] are more likely to allow some extensions to the case $s > 1 + 1/p$.

Another strategy to show that two elements in $W^{s,p}(M,N)$ are $W^{s,p}$ connected is based on the property $P(u)$ defined for any $u \in W^{s,p}(M,N)$ by:

$(P(u))$ The map u is $W^{s,p}$ homotopic to some $\tilde{u} \in C^\infty(M,N)$.

We proceed to explain the interest of this property. Assume that $P(u)$ and $P(v)$ are true, where $u, v \in W^{s,p}(M,N)$, and that $\tilde{u}$ and $\tilde{v}$ are C^0 homotopic. So, there exists $F \in C^\infty([0,1] \times M, N)$ such that $F(0,\cdot) = \tilde{u}$ and $F(1,\cdot) = \tilde{v}$, which implies that $\tilde{u}$ and $\tilde{v}$ are $W^{s,p}$ homotopic. Finally, u and v are $W^{s,p}$ homotopic. This shows the importance of the property P.

Theorem 4.5 *Each $u \in W^{s,p}(M,N)$ satisfies $P(u)$ when*
a) $sp \geq \dim M$,
b) $0 < sp < 2, 0 < s < 1 + 1/p$,
c) $\dim M = 2, 0 < s < 1 + 1/p$,
d) $M = S^m, 0 < s < 1 + 1/p$,
e) $0 < s < 1 + 1/p, 2 \leq sp$ and M satisfies the $[sp] - 1$ extension property with respect to N,
f) $0 < s < 1 + 1/p, 2 \leq sp < \dim M$ and $\pi_i(N) = 0$ for $[sp] \leq i \leq \dim M - 1$.

The case $sp \geq \dim M$ can be handled as in the appendix of [20]. If $0 < sp < 2$, then Theorem 4.1 shows that u can be connected to a constant map. The case $\dim M = 2$ is a consequence of a) and b). When $M = S^m$, we can even show that $W^{s,p}(S^m, N)$ is path-connected if $sp < m$ (see section 4.5). The statement f) follows from e) (see [40], Remark 5.1). For the meaning of the "$[sp] - 1$ extension property with respect to N", one should refer to [40] or to section 4.9. Roughly speaking, this means that for any smooth triangulation of M, and any continuous map $f : M^{[sp]} \to N$, we may find a continuous extension of $f|_{M^{[sp]-1}}$ to the whole M. Unfortunately, it is not the case that for any M, N, s, p, each $u \in W^{s,p}(M,N)$ satisfies $P(u)$, (see [40], Corollary 1.5.).

Remark 4.1 *In the above results, we have often assumed that $s < 1 + 1/p, 1 < sp$. This is closely linked to the strategy of our proofs because we glue several maps in $W^{s,p}(M,N)$ together. Let $u_1 \in W^{s,p}(\Omega_1)$ and $u_2 \in W^{s,p}(\Omega_2)$, where Ω_1, Ω_2 are two Lipschitz open subsets of $\mathbb{R}^d$ such that*

$$\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \subset \partial\Omega_1 \cap \partial\Omega_2,$$

and $\Omega := \Omega_1 \cup \Omega_2 \cup \Gamma$ is a Lipschitz open set. Since $1 < sp$, we can define the traces of u_1, u_2. Assume that $tr u_1|_\Gamma = tr u_2|_\Gamma$. Then, the map u defined by

$$u(x) = \begin{cases} u_1(x) & \text{when } x \in \Omega_1, \\ u_2(x) & \text{when } x \in \Omega_2 \end{cases}$$

belongs to $W^{s,p}(\Omega)$ when $s < 1 + 1/p$. In contrast, nothing can be said when $s \geq 1 + 1/p$.

Note that when $sp = 1$, we can not glue maps in $W^{s,p}$ any more, since traces are not defined. However, there is a way to overcome this difficulty (see [20], Appendix B and also section 4.2.2). Finally, when $sp < 1$, maps can be glued without any trace compatibility conditions.

Remark 4.2 *To simplify the presentation, we have assumed that M is boundaryless. Nevertheless, all the results above can be generalized to the case when M has a boundary (see [20], Remark 2.1 and [39], section 4).*

Remark 4.3 *Lemma 4.21 below and Theorem 4.4 show that there exists $\eta > 0$ such that for any $f, g \in W^{s,p}(M,N)$, if $||f-g||_{W^{s,p}(M,N)} < \eta$, then f and g are $W^{s,p}$ homotopic. Hence connected components coincide with path-connected components.*

The following section is the technical core of the article: it enumerates some variations of the technique 'filling a hole', a phrase coined by Brezis and Li [20]. Sections 4.3 and 4.4 present some consequences of this technique which allow us to generalize in section 4.5 the results of [20] ; that is, Theorem 4.1 and Theorem 4.5 d). In section 4.6 and section 4.7 , we recall and adapt some results of [40] which prepare the proof of Theorem 4.4 in section 4.8. In the final section, the corollaries of this theorem, namely Theorem 4.2, Theorem 4.3 b) and Theorem 4.5 e) are proved.

We now introduce some notations: In $\mathbb{R}^d$, B^d (or B when no confusion may arise) denotes the unit ball centered at $0, S^d$ (or S) its boundary, $B_r^d(x) := rB + x, S_r^d(x) := rS + x$ and $B_r = rB$, $S_r = rS$. We will use the convention that all the constants are denoted by the same letter C.

When X is a topological space and $u, v \in X$, we write $u \sim_X v$ to signify the fact that there exists $H \in C^0([0,1], X)$ such that $H(0) = u$ and $H(1) = v$. We abbreviate this notation writing $u \sim_{s,p} v$ when u and v are $W^{s,p}$ homotopic; similarly, $u \sim v$ means that u and v are C^0 homotopic.

Whenever $s \in (1, 1+1/p)$, we denote $\sigma := s - 1$.

For any k dimensional Lipschitz manifold D embedded in $\mathbb{R}^n$ and any measurable function f, we denote

$$[f]_{W^{\sigma,p}(D)} := \left(\int_D d\mathcal{H}^k(x) \int_D d\mathcal{H}^k(y) \frac{|f(x) - f(y)|^p}{|x-y|^{n+\sigma p}} \right)^{1/p}.$$

The set $W^{s,p}(M)$ denotes either $W^{s,p}(M, \mathbb{R})$ or $W^{s,p}(M, \mathbb{R}^l)$. This will be clear from the context.

4.2 Filling a hole

The technique 'Filling a hole' appears in [20], Proposition 1.3. We will first generalize it to our context. This will be useful in adapting other tools from [20], such as 'Bridging a map' (see Section 4.3) and 'Opening a map' (see Section 4.4). This will allow us to avoid analytical proofs devised in [20] which elude us in the context of fractional Sobolev spaces.

In this section, the underlying Euclidean space is $\mathbb{R}^n$.

4.2.1 The main result

In this subsection, we prove the following generalization of Lemma D.1 in [14]:

Lemma 4.1 *Let $0 < s < 2, sp < n$ and $u \in W^{s,p}(S)$. Then, the map $\tilde{u}(x) := u(x/|x|)$ belongs to $W^{s,p}(B)$ and we have*

$$||\tilde{u}||_{W^{s,p}(B)} \leq C||u||_{W^{s,p}(S)}. \tag{4.1}$$

Proof: We first prove that $\tilde{u} \in L^p(S)$:

$$\int_B |\tilde{u}(x)|^p \, dx = \int_S |u(\theta)|^p d\theta \int_0^1 r^{n-1} \, dr = 1/n||u||^p_{L^p(S)}.$$

We consider three cases: $s = 1, s > 1$ and $s < 1$. When $s = 1$, we have:

$$\int_B |D\tilde{u}(x)|^p \, dx \leq C \int_S |Du(\theta)|^p d\theta \int_0^1 r^{n-1-p} \, dr \leq C||Du||^p_{L^p(S)},$$

since $p < n$.

When $s \in (1, 2)$, we claim that

$$I := \int_B dx \int_B dy \frac{|D\tilde{u}(x) - D\tilde{u}(y)|^p}{|x - y|^{n+\sigma p}} < +\infty.$$

We denote $f(x) := x/|x|$. We have

$$Df(x) = \frac{1}{|x|} Id - \frac{x \otimes x}{|x|^3} \quad \text{, where } x \otimes x = (x_i x_j)_{(i,j) \in [|1,n|]^2},$$

so that $|Df(x)| \leq C/|x|$ and

$$|Df(x) - Df(y)| \leq C \frac{|x - y|}{|x||y|}. \tag{4.2}$$

(Indeed, note that $Df(\lambda x) = x/\lambda$ and $Df(Rx) = R(Df(x))R^{-1}$ for any $\lambda > 0, R \in O(n)$. Hence, we can assume that

$$x = (1, 0, ..0) \quad \text{and} \quad y = (r \cos\theta, r \sin\theta, 0, .., 0).$$

Then, (4.2) can be easily shown).

Writing

$$\begin{aligned} |D\tilde{u}(x) - D\tilde{u}(y)| \leq & |Du(x/|x|) - Du(y/|y|)||Df(x)| \\ & + |Du(y/|y|)||Df(x) - Df(y)|, \end{aligned} \tag{4.3}$$

we find $I \leq C(I_1 + I_2)$ with

$$I_1 := \int_S d\theta \int_S d\tau |Du(\theta) - Du(\tau)|^p \int_{r=0}^1 dr \int_{t=0}^1 \frac{r^{n-1-p} t^{n-1}}{|r\theta - t\tau|^{n+\sigma p}} \, dt,$$

$$I_2 := \int_B dx \int_B dy |Du(y/|y|)|^p \frac{|x - y|^p}{|x|^p |y|^p |x - y|^{n+\sigma p}}.$$

We claim that whenever $\theta \neq \tau$,

$$J := \int_{r=0}^1 dr \int_{t=0}^1 \frac{r^{n-1-p} t^{n-1}}{|r\theta - t\tau|^{n+\sigma p}} \, dt \leq \frac{C}{|\theta - \tau|^{n-1+\sigma p}}. \tag{4.4}$$

Indeed, after making the change of variable $t \to \lambda := t/r$, we get

$$J \leq \int_{r=0}^{1} r^{n-1-sp}\,dr \int_{\lambda=0}^{\infty} \frac{\lambda^{n-1}}{|\theta-\lambda\tau|^{n+\sigma p}}\,d\lambda$$

$$\leq C \int_{\lambda=0}^{\infty} \frac{\lambda^{n-1}}{|\theta-\lambda\tau|^{n+\sigma p}}\,d\lambda \qquad \text{(since } sp<n\text{)}$$

$$\leq C(\int_{\lambda=0}^{2} \frac{d\lambda}{|\theta-\lambda\tau|^{n+\sigma p}} + \int_{2}^{\infty} \frac{\lambda^{n-1}}{\lambda^{n+\sigma p}}) \leq C(\int_{\lambda=0}^{2} \frac{d\lambda}{|\theta-\lambda\tau|^{n+\sigma p}} + 1).$$

Now, consider the 2 plane generated by θ and τ. In this plane, θ and τ belongs to S^1, so that they can be written $\theta = e^{i\alpha}, \tau = e^{i\beta}, \alpha, \beta \in (-\pi, \pi]$. Hence, with $\gamma := \beta - \alpha$,

$$|\theta-\lambda\tau|^2 = |\lambda - e^{i\gamma}|^2 = (\lambda-\cos\gamma)^2 + \sin^2\gamma.$$

The change of variable $\mu := (\lambda - \cos\gamma)/\sin\gamma$, (when $\sin\gamma \neq 0$) yields

$$\int_{\lambda=0}^{2} \frac{d\lambda}{|\theta-\lambda\tau|^{n+\sigma p}} \leq \frac{1}{(\sin\gamma)^{n-1+\sigma p}} \int_{\mathbb{R}} \frac{d\mu}{(1+\mu^2)^{(n+\sigma p)/2}} \leq \frac{C}{(\sin\gamma)^{n-1+\sigma p}}.$$

Moreover,

$$|\theta-\tau|^2 = 2(1-\cos\gamma) = 4\sin^2(\gamma/2)$$

and the map $\gamma \to \dfrac{\sin(\gamma/2)}{\sin\gamma}$ is bounded near 0, say for $|\gamma| \leq \pi/4$. This shows that

$$\int_{\lambda=0}^{2} \frac{d\lambda}{|\theta-\lambda\tau|^{n+\sigma p}} \leq \frac{C}{|\theta-\tau|^{n-1+\sigma p}}$$

when $|\beta-\alpha| \leq \pi/4$. On the other hand, this inequality is trivially true when $|\beta-\alpha| \geq \pi/4$ (by increasing C if necessary). This proves (4.4) and implies that

$$I_1 \leq C \int_S d\theta \int_S d\tau \frac{|Du(\theta)-Du(\tau)|^p}{|\theta-\tau|^{n-1+\sigma p}} = C[Du]^p_{W^{\sigma,p}(S)}.$$

We proceed to estimate I_2. We have

$$I_2 \leq \int_B |Du(y/|y|)|^p\,dy \int_{\mathbb{R}^n} \frac{dx}{|x|^p|y|^p|y-x|^{n+(\sigma-1)p}}$$

$$=: \int_B |Du(y/|y|)|^p K(y)\,dy.$$

Clearly, for any $y \neq 0$, $K(y) < \infty$ (since $p<n$), $K(y)$ depends only on $|y|$ and $K(\lambda y) = K(y)/\lambda^{sp}$. Thus, $K(y) = C/|y|^{sp}$. This shows that $I_2 \leq C||Du||^p_{L^p(S)}$. Moreover, we have established (4.1) when $s \in (1,2)$.

When $s \in (0,1)$, the calculation is easier, and is very similar to the treatment of I_1. The lemma is proved.

□

The same proof yields:

Corollary 4.1 *Let $0<s<2, sp<n$ and $u \in W^{s,p}(S)$. Then, $\tilde{u}(x) := u(x/|x|)$ belongs to $W^{s,p}_{loc}(\mathbb{R}^n)$.*

4.2.2 Filling a hole continuously

Consider a smooth bounded open set Ω in $\mathbb{R}^n$ and denote by Γ its boundary. There exists $\epsilon > 0$ such that the ϵ tubular neighborhood of Γ :

$$U_\epsilon := \{x \in \Omega : \mathrm{dist}\,(x, \Gamma) < \epsilon\}$$

can be parametrized by:

$$\Phi : (x', r) \in \Gamma \times (0, \epsilon) \mapsto x' + r\nu(x'),$$

where $\nu(x')$ denotes the inner unit normal to Γ at x'. We also introduce the nearest point projection $\pi : U_\epsilon \to \Gamma$. Hence, for any $x \in U_\epsilon$, we have $\Phi^{-1}(x) = (\pi(x), \mathrm{dist}\,(x, \Gamma))$. Finally, we denote $\Gamma_r := \Phi(\Gamma \times \{r\})$.

Note that for any measurable function $u : \mathbb{R}^n \to \mathbb{R}$, defined almost everywhere, it makes sense to define its restriction $u|_{\Gamma_r}$ to Γ_r, for almost every $r \in (0, \epsilon)$. When $u \in W^{s,p}(\mathbb{R}^n)$ with $sp > 1$, this restriction is equal to the trace of $u : \mathrm{tr}\, u|_{\Gamma_r}$ for a.e. r. In the special case $sp = 1$, we need a substitute for the trace theory: the *good restrictions*, introduced in [14]. We proceed to present the definition of good restrictions for a map $u \in W^{s,p}(\Omega)$, when $s \in (0, 1), sp = 1$. For a proof of the statements below, see [14].

For each $r \in (0, \epsilon)$, there is at most one function v defined on Γ_r such that the map

$$w_1^r(x) = \begin{cases} u(x) \ \text{ in } \Omega \setminus U_r, \\ v(\Phi(\pi(x), r)) \ \text{ in } \Omega \cap U_r \end{cases}$$

or equivalently, the map

$$w_2^r(x) = \begin{cases} u(x) - v(\Phi(\pi(x), r)) \ \text{ in } \Omega \setminus U_r, \\ 0 \ \text{ in } \Omega \cap U_r \end{cases}$$

belongs to $W^{s,p}(\Omega)$. Moreover, for a.e. $r \in (0, \epsilon)$, the function $v := u|_{\Gamma_r}$ has the property that $w_1^r, w_2^r \in W^{s,p}(\Omega)$. In fact, a necessary and sufficient condition for this property to hold is that $v \in W^{s,p}(\Gamma_r)$ and

$$\int_\Gamma d\mathcal{H}^{n-1}(x') \int_r^\epsilon dt \frac{|v(\Phi(x', r)) - u(\Phi(x', t))|^p}{(t - r)} < \infty.$$

For these values of r, we say that v is the inner good restriction of u to Γ_r. Similarly, we may define an outer good restriction. If v is both an inner and an outer good restriction, we call it a good restriction.

In particular, $u|_{\Gamma_r}$ is a good restriction if and only if

i) $u|_{\Gamma_r} \in W^{s,p}(\Gamma_r)$,

ii) $\displaystyle\int_\Gamma d\mathcal{H}^{n-1}(x') \int_0^\epsilon dt \frac{|u(\Phi(x', r)) - u(\Phi(x', t))|^p}{|t - r|} < \infty.$

Assume that Γ can be written as a finite union of subsets Γ^i which are open in Γ and such that i), ii) are true for each Γ^i instead of Γ. Then i), ii) are true for Γ. This shows that 'being a good restriction' is a *local* condition.

We will often use the following well-known consequence of the Fubini's Theorem:

Lemma 4.2 *Let $s \in (0,2)$ and $u \in W^{s,p}(\Omega)$. Then for a.e. $r \in (0,\epsilon)$,*
i) when $sp > 1$, the trace $tr u|_{\Gamma_r}$ coincides with $u|_{\Gamma_r}$ and belongs to $W^{s,p}(\Gamma_r)$,
ii) when $sp = 1$, $u|_{\Gamma_r}$ is a good restriction of u to Γ_r, (in particular, $u|_{\Gamma_r} \in W^{s,p}(\Gamma_r)$),
iii) when $sp < 1$, the restriction of u to Γ_r belongs to $W^{s,p}(\Gamma_r)$.

Such an r will be called '*good*'. We will also say that Γ_r is '*good for u*'.

In the following lemma, the set Ω is B_2, so that Γ_r is the sphere of radius $2-r$.

Lemma 4.3 *Let $0 < s < 1+1/p, 0 < sp < n$. Let $u \in W^{s,p}(B_2, N)$ and assume that S is good for u. For any $t \in [0,1)$, let*

$$u^t(x) = \begin{cases} u(x/(1-t)) & \textit{when } |x| \leq 1-t, \\ u(x/|x|) & \textit{when } 1-t \leq |x| \leq 1, \\ u(x) & \textit{when } 1 \leq |x| \leq 2 \end{cases}$$

and

$$u^1(x) = \begin{cases} u(x/|x|) & \textit{when } |x| \leq 1, \\ u(x) & \textit{when } 1 \leq |x| \leq 2. \end{cases}$$

Then,

$$t \in [0,1] \to u^t \in W^{s,p}(B_2, N)$$

is continuous and $u^t(x) = u(x)$ for any $t \in [0,1]$ and any $1 \leq |x| \leq 2$.

Proof: Consider the maps

$$v^t(x) = \begin{cases} u(x/(1-t)) & \text{when } |x| \leq 1-t, \\ u(x/|x|) & \text{when } 1-t \leq |x| \leq 2 \end{cases}$$

and $v^1(x) = u(x/|x|)$. To prove Lemma 4.3, it is enough to show that $v^t \in C^0([0,1], W^{s,p}(B_2,N))$ since $u^t = v^t + z$ where z is defined by:

$$z(x) = \begin{cases} 0 & \text{when } |x| \leq 1, \\ u(x) - u(x/|x|) & \text{when } 1 \leq |x| \leq 2. \end{cases}$$

(The map z belongs to $W^{s,p}$ since S is good for u.)

Consider first the case $sp > 1$. Then, Lemma 4.3 is essentially Lemma D.2 in [14] : condition $s < 1$ is replaced by $s < 1 + 1/p$ in our case.

Let

$$\tilde{v}(x) := \begin{cases} u(x) & \text{when } |x| \leq 1, \\ u(x/|x|) & \text{when } 1 \leq |x|. \end{cases}$$

Then $\tilde{v}$ belongs to $W^{s,p}_{\text{loc}}(\mathbb{R}^n)$. We have $v^t(x) = \tilde{v}(x/(1-t))$. This shows that $t \in [0,1) \mapsto v^t \in W^{s,p}(B_2,N)$ is continuous. Thus, there remains to show that v^t converges to v^1 when $t \to 1^-$. By Corollary 4.1, $v^1 \in W^{s,p}_{\text{loc}}(\mathbb{R}^n)$. Let $g := \tilde{v} - v^1$. Then, $g \in W^{s,p}(\mathbb{R}^n)$ because $g(x) = 0$ when $|x| \geq 1$. Moreover, $v^t(x) - v^1(x) = g(x/(1-t))$. We easily have

$$[g(\cdot/(1-t))]_{W^{s,p}(\mathbb{R}^n)} = (1-t)^{(n-sp)/p}[g]_{W^{s,p}(\mathbb{R}^n)}.$$

This shows the continuity at $t = 1$.

It remains to consider the case $sp \leq 1$. Though we cannot define the trace anymore, the fact that $r = 1$ is good implies that $\tilde{v} \in W^{s,p}_{\text{loc}}(\mathbb{R}^n), g \in W^{s,p}(\mathbb{R}^n)$. As above, we find that $v^t \to v^1$ in $W^{s,p}(B_2)$.

This completes the proof of the lemma.

□

4.2.3 Filling an annulus continuously

As a corollary of Lemma 4.3, we get the following:

Lemma 4.4 *Let $s \in (0, 1+1/p)$ and $u \in W^{s,p}(B_2)$ such that S is good for u. Then, the map u^t defined by*

$$u^t(x) = \begin{cases} u(x/(1-t/2)) & \text{when } |x| \leq 1-t/2, \\ u(x/|x|) & \text{when } 1-t/2 \leq |x| \leq 1, \\ u(x) & \text{when } 1 \leq |x| \leq 2 \end{cases}$$

belongs to $C^0([0,1], W^{s,p}(B_2))$.

Lemma 4.4 can be immediately generalized to the case when B_2 is replaced by a smooth bounded open convex set Ω containing the origin, with the Euclidean norm replaced by the norm

$$j(x) := \inf\{t > 0 : x/t \in \Omega\}.$$

4.2.4 Filling a cylinder

In this subsection, we pick some $2 \leq k \leq n-1$ and we decompose $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. We also denote $x \in \mathbb{R}^n$ as $(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Let T be the open set in $\mathbb{R}^n$ defined by:

$$T := \{(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x'| < 1\}$$

and $2T := \{2x : x \in T\}$. Then we have:

Lemma 4.5 *Let $0 < s < 2, sp < k$ and $u \in W^{s,p}(\partial T)$. Then, the map $\tilde{u}$ defined by:*

$$\tilde{u}(x', x'') := u(x'/|x'|, x'')$$

belongs to $W^{s,p}(T)$.

Proof: An easy computation shows that

$$||\tilde{u}||_{W^{1,p}(T)} \leq C||u||_{W^{1,p}(\partial T)} \qquad ;$$

this settles the case $s = 1$. When $s \in (1, 2)$, it remains to show that

$$I := \int_T dx \int_T dy \frac{|D\tilde{u}(x) - D\tilde{u}(y)|^p}{|x-y|^{n+\sigma p}} < +\infty.$$

We have $I \leq C(I' + I'')$, where

$$I' := \int_{\mathbb{R}^{n-k}} dx'' \int_{x' \in \mathbb{R}^k, |x'|<1} dx' \int_{y' \in \mathbb{R}^k, |y'|<1} dy' \frac{|D\tilde{u}(x', x'') - D\tilde{u}(y', x'')|^p}{|x'-y'|^{k+\sigma p}},$$

$$I'' := \int_{\mathbb{R}^k, |y'|<1} dy' \int_{x'' \in \mathbb{R}^{n-k}} dx'' \int_{y'' \in \mathbb{R}^{n-k}} dy'' \frac{|D\tilde{u}(y', x'') - D\tilde{u}(y', y'')|^p}{|x''-y''|^{n-k+\sigma p}}.$$

This is a Besov's type inequality (see [1] or [4]).

We first prove that $I'' \leq C||Du||^p_{W^{\sigma,p}(\partial T)}$. Using the fact that $p < n$, we have

$$I'' \leq \int_{|y'|<1} dy' \frac{1}{|y'|^p} \int_{\mathbb{R}^{n-k}} dx'' \int_{\mathbb{R}^{n-k}} dy'' \frac{|Du(y'/|y'|, x'') - Du(y'/|y'|, y'')|^p}{|x'' - y''|^{n-k+\sigma p}}$$

$$\leq C \int_{S^{k-1}} d\theta \int_{\mathbb{R}^{n-k}} dx'' \int_{\mathbb{R}^{n-k}} dy'' \frac{|Du(\theta, x'') - Du(\theta, y'')|^p}{|x'' - y''|^{n-k+\sigma p}},$$

which implies that $I'' \leq C||u||^p_{W^{s,p}(\partial T)}$.

We denote $f(x', x'') := (x'/|x'|, x'')$. We proceed to estimate I' by writing $I' \leq C(I'_1 + I'_2)$ with

$$I'_1 := \int_{\mathbb{R}^{n-k}} dx'' \int_{|x'|<1} dx' \int_{|y'|<1} \frac{|Du(x'/|x'|, x'') - Du(y'/|y'|, x'')|^p}{|x'|^p |x' - y'|^{k+\sigma p}} dy'$$

$$I'_2 := \int_{\mathbb{R}^{n-k}} dx'' \int_{|x'|,|y'|<1} dx'\, dy' \frac{|Du(y'/|y'|, x'')|^p |Df(x', x'') - Df(y', x'')|^p}{|x' - y'|^{k+\sigma p}};$$

this follows from (4.3).

We can prove that $I'_2 \leq C||Du||^p_{L^p(\partial T)}$ exactly as we estimated I_2 in the proof of Lemma 4.1.

On the other hand, we find that

$$I'_1 = \int_{\mathbb{R}^{n-k}} dx'' \int_{|x'|<1} dx' \int_{|y'|<1} dy' \frac{|Du(x'/|x'|, x'') - Du(y'/|y'|, x'')|^p}{|x'|^p |x' - y'|^{k+\sigma p}}$$

$$= \int_{\mathbb{R}^{n-k}} dx'' \int_{S^{k-1}} d\theta \int_{S^{k-1}} d\tau |Du(\theta, x'') - Du(\tau, x'')|^p \int_0^1 \int_0^1 \frac{r^{n-1} t^{n-1}}{r^p |r\theta - t\tau|^{k+\sigma p}}$$

$$\leq C \int_{\mathbb{R}^{n-k}} dx'' \int_{S^{k-1}} d\theta \int_{S^{k-1}} d\tau \frac{|Du(\theta, x'') - Du(\tau, x'')|^p}{|\theta - \tau|^{k-1+\sigma p}},$$

(here, we use $\displaystyle\int_{r=0}^1 dr \int_{t=0}^1 dt \frac{r^{n-1-p} t^{n-1}}{|r\theta - t\tau|^{k+\sigma p}} \leq \frac{C}{|\theta - \tau|^{k-1+\sigma p}}$, see the proof of (4.4)).

From the last inequality, we easily obtain $I'_1 \leq C||u||^p_{W^{s,p}(\partial T)}$, which gives the required result when $s \in (1, 2)$. When $s \in (0, 1)$, the calculation is easier and we omit it. Lemma 4.5 is proved.

□

Lemma 4.5 implies the following (exactly as Lemma 4.1 implied Lemma 4.3):

Lemma 4.6 *Let $0 < s < 1 + 1/p$, $sp < k$ and $u \in W^{s,p}(2T)$ such that ∂T is good for u. Then the map u^t defined by*

$$u^t(x) := \begin{cases} u(x'/(1-t), x'') & \text{when } |x'| \leq 1 - t, \\ u(x'/|x'|, x'') & \text{when } 1 - t \leq |x'| \leq 1, \\ u(x', x'') & \text{when } 1 \leq |x'| \leq 2 \end{cases}$$

belongs to $C^0([0, 1], W^{s,p}(2T))$.

4.3 'Bridging' of maps

4.3.1 The case $n = 2$

Consider the square

$$\Omega := \{x = (x_1, x_2) : |x_1| < 20, \quad |x_2| < 20\}$$

and let $u \in W^{s,p}(\Omega, N)$.

We assume that u is constant, say Y_0, in the region $Q^+ \cup Q^-$ where

$$Q^+ = \{x = (x_1, x_2) : |x_1| < 20, \quad 1 < x_2 < 20\}$$

and

$$Q^- = \{x = (x_1, x_2) : |x_1| < 20, \quad -20 < x_2 < -1\}.$$

The following lemma corresponds to [20], Proposition 1.2.

Lemma 4.7 *If* $0 < s < 1 + 1/p, sp < 2$, *then there exists* $u^t \in C^0([0,1], W^{s,p}(\Omega, N))$ *such that*

$$u^0 = u,$$
$$u^t(x) = u(x) \; \forall t \in [0,1], \quad \forall x \notin (-5,5) \times (-1,1),$$
$$u^1(x) = Y_0 \; \forall x \in (1, 3/2) \times (-20, 20).$$

Proof: First, choose two circles C_1, C_2 with the same radius larger than $2/\sqrt{3}$, centered on the line $\{x = (x_1, x_2) : x_2 = 0\}$ such that the center of C_1 belongs to C_2. This implies that C_1 and C_2 intersects at two points which belongs to Q^+ and Q^-. Moreover, we require that C_1 and C_2 are good for u. Without loss of generality, we may assume that C_1 is centered at $(0,0)$ and that C_2 is centered at $(2,0)$, their common radius being 2. Now, by filling the hole inside C_1 (see Lemma 4.3), we can link u to some u_1 which is equal to u outside C_1 and which is equal to Y_0 on the set $\{(x_1, x_2) : |x_2| \geq |x_1|/\sqrt{3}\}$.

We claim that C_2 is still good for u_1. In fact, in the subset of C_2 where u has been changed, u_1 is equal to Y_0 and when $sp > 1$, the trace of u on $C_2 \cap \{x : x_1 \leq 2\}$ is equal to Y_0. This settles the cases $sp > 1$. The case $sp < 1$ is obvious. When $sp = 1$, it remains to prove that the constant map equal to Y_0 is a good restriction for u to $C_2 \cap \{x : x_1 \leq 2\}$ (since the concept of good restrictions is local). But this is a mere consequence of Lemma 4.8 below. The claim is proved.

Finally, by filling the hole inside C_2, we can connect u_1 to some u_2 which is equal to u_1 outside C_2 while inside C_2, u_2 is equal to Y_0 except on the domain $\{(x_1, x_2) : x_1 > 2+\sqrt{3}|x_2|\}$. In particular, u_2 is equal to u on $\{(x_1, x_2) : |x_1| > 4\}$ and is equal to Y_0 on

$$Q^+ \cup Q^- \cup \{(x_1, x_2) : 0 < x_1 < 2\}.$$

This completes the proof of the lemma.

□

Lemma 4.8 *Let* $sp = 1$ *and* $u \in W^{s,p}((-1,1)^2)$ *such that* $u = Y_0$ *on* $\{x : |x_1| < |x_2|\}$. *Then the constant map equal to* Y_0 *on the line* $D := \{x_1 = 0\}$ *is a good restriction of* u *to* D.

Proof: It is sufficient to prove that

$$I := \int_{-1}^{1} dx_2 \int_{-1}^{1} \frac{|u(x_1, x_2) - Y_0|^p}{|x_1|} \, dx_1 < \infty.$$

Since N is compact, there exists $C > 0$ such that $|u(x_1, x_2) - Y_0|^p \leq C$ for any (x_1, x_2). Then the lemma follows from the fact that:

$$\begin{aligned} I &= \int_{-1}^{1} dx_2 \int_{|x_2| \leq |x_1| \leq 1} \frac{|u(x_1, x_2) - Y_0|^p}{|x_1|} \, dx_1 \\ &\leq C \int_{-1}^{1} dx_2 \int_{|x_2|}^{1} \frac{dx_1}{|x_1|} \leq C. \end{aligned}$$

□

4.3.2 The case $n \geq 2$

We work in $\mathbb{R}^n, n \geq 2$ and we distinguish some special variables. For $0 \leq l \leq n-2$, we write

$$x = (x_1', x'', x_2')$$

where $x_1' = x_1, x_2' = (x_{n-l}, .., x_n)$ and $x'' = (x_2, .., x_{n-l-1})$ (when $l = n-2$, we omit x''). We also write $x' = (x_1', x_2')$. Let

$$\Omega := \{(x_1', x'', x_2') : |x_1'| < 20, |x''| < 20, |x_2'| < 20\}.$$

Set $k := l + 2$.

Lemma 4.9 *Assume that $0 < s < 1 + 1/p, sp < k$ and $u \in W^{s,p}(\Omega, N)$ with $u(x) = Y_0$ for any $x \in \Omega$ such that $1 < |x_2'|$, for some $Y_0 \in N$. Then there exists $u^t \in C^0([0,1], W^{s,p}(\Omega, N))$ such that $u^0 = u, u^t(x) = u(x)$ for any $0 \leq t \leq 1$ and any x outside $\{x : |x| < 15\}$ and $u^1(x) = Y_0$ for any $x, |(x_1', x'')| < 1/8$.*

Proof: If $k = n$, then the proof is exactly the same as in the previous subsection (except that circles are replaced by n dimensional balls). Hence, we may assume that $k < n$. Let $\delta : \mathbb{R}^{n-k} \to \mathbb{R}^k$ be a smooth function to be chosen later. We define the cylinder C_1 by

$$C_1 := \{x = (x_1', x'', x_2') : |x' - \delta(x'')| = a\}$$

and the tube T_1 by

$$T_1 := \{x = (x_1', x'', x_2') : |x' - \delta(x'')| < a\},$$

for some $a > 1$ to be determined below. We may choose a and δ such that:
i) when $|x''| < 2$, we have $\delta(x'') = 0$,
ii) when $|x''| \geq 4$, we have $x \in T_1 \Rightarrow |x_2'| > 1$,
iii) C_1 is good for u.

Note that C_1 can be chosen as a smooth deformation of a *straight* cylinder as defined in subsection 4.2.4. Note also that even if $C_1 \cap \Omega$ is a *finite* cylinder (contrary to those of subsection 4.2.4), the *ends* of this cylinder are contained in a domain where u is equal to the constant Y_0, where 'nothing happens'. Hence, we can apply Lemma 4.6 to C_1 : u can be connected to some $\bar{u}$ which equals Y_0 on $\{x \in \Omega : |x''| < 2, |x_2'| \geq |x_1'|/\sqrt{a^2-1}\}$.

The computation in the proof of Lemma 4.8 yields easily that $\bar{u}$ has a good restriction (equal to Y_0) on the set $\{|x''| < 2, x_1' = 0\}$. This implies that the map:

$$w(x_1', x'', x_2') := \begin{cases} 0 \text{ when } x_1' \leq 0, \\ \bar{u}(x_1', x'', x_2') - Y_0 \text{ when } x_1' \geq 0 \end{cases}$$

belongs to $W^{s,p}(\Omega_0)$, where $\Omega_0 := \{x \in \Omega : |x''| < 2\}$.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth function which vanishes on $\{t : |t| \geq 2\}$, which is equal to 1 on $\{t : |t| \leq 1\}$ and such that $|\rho'| \leq 2$. Then we define

$$\Xi_t(x_1', x'', x_2') := (x_1' - \frac{t\rho(2|x''|^2)\rho(2x_1')}{8}, x'', x_2').$$

The map Ξ_t is a smooth diffeomorphism of $\mathbb{R}^n$ which maps Ω_0 onto Ω_0.

By the *diffeomorphism property* in $W^{s,p}$ (see [86]), there exists $C > 0$ such that for any $t \in [0,1]$, and any $g \in W^{s,p}(\Omega_0)$, we have

$$||g \circ \Xi_t||_{W^{s,p}(\Omega_0)} \leq C||g||_{W^{s,p}(\Omega_0)}.$$

Let $\epsilon > 0$. Then there exists $z \in C^\infty(\bar{\Omega}_0)$ such that $||z - w||_{W^{s,p}(\Omega_0)} < \epsilon$. Hence, for any $t, s \in [0,1]$,

$$||w \circ \Xi_t - w \circ \Xi_s||_{W^{s,p}(\Omega_0)} \leq ||w \circ \Xi_t - z \circ \Xi_t||_{W^{s,p}(\Omega_0)} + ||z \circ \Xi_t - z \circ \Xi_s||_{W^{s,p}(\Omega_0)}$$

$$+||z \circ \Xi_s - w \circ \Xi_s||_{W^{s,p}(\Omega_0)} \leq C||z - w||_{W^{s,p}(\Omega_0)} + ||z \circ \Xi_t - z \circ \Xi_s||_{W^{s,p}(\Omega_0)}$$

$$\leq C\epsilon + ||z \circ \Xi_t - z \circ \Xi_s||_{W^{s,p}(\Omega_0)}.$$

Since the last term goes to 0 when $|s - t| \to 0$, the map $t \to w \circ \Xi_t$ belongs to $C^0([0,1], W^{s,p}(\Omega_0))$.

Similarly we may define

$$\tilde{w}(x_1', x'', x_2') := \begin{cases} \bar{u}(x_1', x'', x_2') - Y_0 \text{ when } x_1' \leq 0, \\ 0 \text{ when } x_1' \geq 0 \end{cases}$$

and

$$\tilde{\Xi}_t(x_1', x'', x_2') := (x_1' + \frac{t\rho(2|x''|^2)\rho(2x_1')}{8}, x'', x_2').$$

As above, $\tilde{w} \circ \tilde{\Xi}_t \in C^0([0,1], W^{s,p}(\Omega_0))$. This yields

$$w \circ \Xi_t + \tilde{w} \circ \tilde{\Xi}_t \in C^0([0,1], W^{s,p}(\Omega_0)).$$

If we denote by v^t the map $w \circ \Xi_t + \tilde{w} \circ \tilde{\Xi}_t + Y_0$, we have $v^t =$

$$\begin{cases} \bar{u}(x_1' + t\rho(2|x''|^2)\rho(2x_1')/8, x'', x_2') \text{ when } x_1' \leq -t\rho(2|x''|^2)\rho(2x_1')/8, \\ \\ Y_0 \text{ when } -t\rho(2|x''|^2)\rho(2x_1')/8 \leq x_1' \leq t\rho(2|x''|^2)\rho(2x_1')/8, \\ \\ \bar{u}(x_1' - t\rho(2|x''|^2)\rho(2x_1')/8, x'', x_2') \text{ when } t\rho(2|x''|^2)\rho(2x_1')/8 \leq x_1'. \end{cases}$$

Note in particular that $v^t = \bar{u}$ when $|x''| > 1$ or $|x_1'| > 1$. Hence we can extend v^t by $\bar{u}$ on Ω and we still have $v^t \in C^0([0,1], W^{s,p}(\Omega))$. Finally, $v^t = Y_0$ when $|x''| < 1/\sqrt{2}$ and $|x_1'| \leq t/8$. This completes the proof of the lemma.

□

4.4 Opening of Maps

Lemma 4.10 *Let $0 < s < 1 + 1/p$ and $u \in W^{s,p}(B_{10}, N)$. Then, there exists $u^t \in C^0([0,1], W^{s,p}(B_{10}, N))$ such that $u^0 = u, u^1 = Y_0$ on an open subset of B_5 for some $Y_0 \in N$ and $u^t = u$ on $B_{10} \setminus B_9, 0 \leq t \leq 1$.*

Proof: We first introduce the concept of *smooth cubes.* A smooth cube is simply a cube with smooth corners, or equivalently, a sphere with faces. Formally, a smooth open set G of $\mathbb{R}^n$ will be called a smooth cube of side R if it is a smooth convex set G which satisfies:

$$\cup_{i=1}^{n}\{(x_1, .., x_n) : |x_i| < R, |x_j| < 4R/5 \quad \forall j \neq i\} \subset G \subset (-R, R)^n.$$

For such a set G, we define the i^{th} face:

$$F_i := \{(x_1, ..x_n) : x_i = R, |x_j| < 4R/5\}.$$

For any $i = 1, ..n$, let

$$G_i := \{tx : x \in F_i, t \in (1/5, 1)\}.$$

The set G is a smooth convex set, so that the technique of 'filling an annulus' (see Lemma 4.4) applies. More precisely, consider some $v \in W^{s,p}(\mathbb{R}^n)$ such that ∂G is good for v. Then v can be connected to a map $w \in W^{s,p}(\mathbb{R}^n)$ which is equal to v on $\mathbb{R}^n \setminus G$ and which satisfies

$$w(tx) = v(x) \quad \forall tx \in G_i.$$

Returning to the proof of Lemma 4.10, let $v \in W^{s,p}(B_{10})$ and G be a smooth cube of side R such that $G \subset B_5$ and ∂G is good for v. Assume that $v|_{F_i}(x_1, .., x_n)$ does not depend on $x_1, .., x_{i-1}$. By this, we mean that for $\mathcal{H}^{n-i+1}$ a.e. $x_i, .., x_n \in \mathbb{R}^{n-i+1}$, the map $(x_1, ..x_{i-1}) \in \mathbb{R}^{i-1} \to \chi_{F_i}(x)v(x)$ is $\mathcal{H}^{i-1}$ a.e. constant. Then on $G_i, w(tx) = v(x)$ (with $x \in F_i, t \in (1/5, 1)$), does not depend neither on $x_1, .., x_{i-1}$ nor on t.

Consider the map

$$\phi_i : tx \in G_i \mapsto \sum_{j \neq i} \frac{5x_j}{4R} e_j + \frac{5t - 3}{2} e_i \in (-1, 1)^n.$$

Here (e_k) denotes the canonical basis of $\mathbb{R}^n$. Observe that ϕ_i^{-1} is a smooth diffeomorphism from $[-1, 1]^n$ onto $\bar{G}_i$. Then, $w \circ \phi_i^{-1} \in W^{s,p}((-1, 1)^n)$ and does not depend on $x_1, .., x_i$.

We now prove the lemma by induction: We claim that for each $1 \leq k \leq n, u$ can be connected to some $u_k \in W^{s,p}(B_{10})$ such that $u_k = u$ outside B_9 and such that there exists a smooth diffeomorphism ψ_k from $[-1, 1]^n$ into B_5 such that $u_k \circ \psi_k$ does not depend on $x_1, .., x_k$ on $(-1, 1)^n$.

For $k = 1$, select a smooth cube $G \subset B_5$ such that ∂G is good for u. Then as explained above, we can connect u to some u_1 which is equal to u on $B_{10} \setminus G$ and such that $u_1(tx) = u(x)$ for any $x \in F_1, t \in (1/5, 1)$. Then $u_1 \circ \phi_1^{-1}$ belongs to $W^{s,p}((-1, 1)^n)$ and does not depend on x_1. We can choose $\psi_1 = \phi_1^{-1}$.

Assume the claim is true up to k. We can select a smooth cube G inside $(-1, 1)^n$, such that ∂G is good for $u_k \circ \psi_k$ and $u_k \circ \psi_k$ does not depend on

$x_1,..,x_k$ on G. Then, as explained previously, we can connect $u_k \circ \psi_k$ to some $w \in W^{s,p}((-1,1)^n)$ such that $w = u_k \circ \psi_k$ on $(-1,1)^n \setminus G$ and $w(tx) = u_k \circ \psi_k(x)$ for any $x \in F_{k+1}, F_{k+1}$ being the $(k+1)^{th}$ face relative to G. Then $w \circ \phi_{k+1}^{-1}$ (ϕ_{k+1} being defined for G) belongs to $W^{s,p}((-1,1)^n)$ and does not depend on $x_1,..,x_{k+1}$. We can choose $\psi_{k+1} = \psi_k \circ \phi_{k+1}^{-1}$ and define

$$u_{k+1}(x) := \begin{cases} u_k(x) & \text{when } \; x \in B_{10} \setminus \psi_k(G), \\ w \circ \psi_k^{-1}(x) & \text{when } \; x \in \psi_k(G). \end{cases}$$

The claim is proved for $k+1$. Finally, we have connected u to a map $u_n \in W^{s,p}(B_{10})$ which is a.e. constant on $\psi_n((-1,1)^n)$, namely an open subset of B_5.

□

4.5 Proof of Theorem 4.1 and Theorem 4.5 c)

The tools 'Connecting constants' and 'Propagation of constants' in [20] can be readily generalized to the case $W^{s,p}$.

Then, the same proof as in [20], Theorem 0.2 shows that $W^{s,p}(M,N)$ is path connected when $sp < 2$; that is, Theorem 4.1. The fact that $W^{s,p}(S^m,N)$ is path-connected when $s \in (0, 1+1/p)$ can be proved as in [20], Proposition 0.1. This shows Theorem 4.5 c).

In the sections below, we assume that $s \in (0, 1+1/p), p \in [1,\infty), 1 < sp$.

We denote by Π_N the nearest point projection onto N, which is defined and smooth on an ϵ_N tubular neighborhood of N :

$$N_{\epsilon_N} := \{x \in \mathbb{R}^l : \operatorname{dist}(x,N) < \epsilon_N\}.$$

4.6 Definition of $[sp-1]$ homotopy

4.6.1 Triangulations and homotopy

We define a rectilinear cell, its dimension, its faces and a rectilinear cell complex as in [73], Chapter 7. In particular, the p skeleton of a rectilinear cell complex K, denoted by K^p, is the collection of all cells having dimension at most p. Any complex considered below is finite. The *polytope* $|K|$ of a complex K is the union of the cells of K. We will use the fact that the boundary $\partial\Delta$ of a simplex Δ can be identified with a complex in an obvious way.

We also introduce some notation. Let Δ be a rectilinear cell, $y \in \operatorname{Int}\Delta$. Then, for any $x \in \Delta$, we set

$$|x|_{y,\Delta} := \inf\{t > 0 : x \in y + t(\Delta - y)\}.$$

This is the usual Minkowski functional of Δ with respect to y. When it is clear what y and Δ are, we simply write $|x|$ instead of $|x|_{y,\Delta}$.

The concepts of smooth maps and immersions on a complex K are defined as in [73], Chapter 8. A smooth immersion which is a homeomorphism onto

M is called a triangulation of M. Actually, the word 'triangulation' is mostly used for the case when K is simplicial. In the general case, we will also use the phrase 'rectilinear cell decomposition'. Each smooth boundaryless manifold M has a triangulation ([73], Theorem 10.6). The proof of this result shows that we can choose a simplicial m dimensional complex K (where m is the dimension of M) such that the polytope $|K|$ is the union of its m simplices. Consider such a simplicial complex and denote by $f : K \to M$ a triangulation. The set $f(\Delta)$ is a Lipschitz domain in M for each cell Δ.

Assume that $u \in W^{s,p}(M)$. Then $u \circ f|_\Delta$ belongs to $W^{s,p}(\Delta)$ for each m cell $\Delta \in K$, because $f|_\Delta$ is a smooth diffeomorphism onto $f(\Delta) \subset M$. Conversely, assume that $u \in L^p(M)$ is such that u belongs to $W^{s,p}(f(\Delta))$ for each m cell $\Delta \in K$. Since $sp > 1$, we can define the trace of u on $\partial f(\Delta)$. Assume that for any m cells $\Delta_1, \Delta_2 \in K$ satisfying $\Delta_1 \cap \Delta_2 \neq \emptyset$, the maps $u|_{f(\Delta_1)}$ and $u|_{f(\Delta_2)}$ have the same trace on $f(\Delta_1 \cap \Delta_2)$. This certainly implies that u belongs to $W^{s,p}(f(\Delta_1 \cup \Delta_2))$ when $s \leq 1$. But this holds true even when $s \in (1, 1 + 1/p)$, because in that case the derivatives of $u|_{f(\Delta_1)}$ and $u|_{f(\Delta_2)}$ belong to $W^{\sigma,p}(f(\Delta_1))$ and $W^{\sigma,p}(f(\Delta_2))$ respectively, with now $\sigma p = (s-1)p < 1$. This implies that the derivatives of u belong to $W^{\sigma,p}(f(\Delta_1 \cup \Delta_2))$. Hence, $u \in W^{s,p}(f(\Delta_1 \cup \Delta_2))$.

The following lemma shows that we can glue homotopies together:

Lemma 4.11 *Let $f : K \to M$ be a smooth triangulation, with m being the common dimension of K and M. Assume that Δ_1 and Δ_2 are two m simplices in K such that $\Delta_1 \cap \Delta_2 = \Sigma$, where Σ is $m-1$ dimensional. Let $F_1 \in C^0([0,1], W^{s,p}(f(\Delta_1)))$, $F_2 \in C^0([0,1], W^{s,p}(f(\Delta_2)))$ and $\forall t \in [0,1]$,*

$$tr\, F_1(t)|_{f(\Sigma)} = tr\, F_2(t)|_{f(\Sigma)}.$$

Then $F \in C^0([0,1], W^{s,p}(f(\Delta_1 \cup \Delta_2)))$ where

$$F(t)(x) = \begin{cases} F_1(t)(x) & \text{when } x \in \Delta_1, \\ F_2(t)(x) & \text{when } x \in \Delta_2. \end{cases}$$

Proof: Let us define the closed subset of $W^{s,p}(f(\Delta_1)) \times W^{s,p}(f(\Delta_2))$:

$$\mathcal{F} := \{(u_1, u_2) \in W^{s,p}(f(\Delta_1)) \times W^{s,p}(f(\Delta_2)) : \mathrm{tr}\, u_1|_{f(\Sigma)} = \mathrm{tr}\, u_2|_{f(\Sigma)}\}.$$

Then the remarks above show that the map: $(u_1, u_2) \in \mathcal{F} \to u \in W^{s,p}(f(\Delta_1 \cup \Delta_2))$ where

$$u(x) = \begin{cases} u_1(x) & \text{when } x \in f(\Delta_1), \\ u_2(x) & \text{when } x \in f(\Delta_2) \end{cases}$$

is well defined.

The Closed Graph Theorem shows that this map is continuous into $W^{s,p}(f(\Delta_1 \cup \Delta_2))$. In particular, there exists $C > 0$ such that for any $(u_1, u_2) \in \mathcal{F}$,

$$||u||_{W^{s,p}(f(\Delta_1 \cup \Delta_2))} \leq C[||u_1||_{W^{s,p}(f(\Delta_1))} + ||u_2||_{W^{s,p}(f(\Delta_2))}]. \tag{4.5}$$

Whence

$$\begin{aligned} ||F(t) - F(t')||_{W^{s,p}(f(\Delta_1 \cup \Delta_2))} &\leq C[||F_1(t) - F_1(t')||_{W^{s,p}(f(\Delta_1))} \\ &\quad + ||F_2(t) - F_2(t')||_{W^{s,p}(f(\Delta_2))}]. \end{aligned}$$

The lemma follows.

□

4.6.2 Definition of $\mathcal{W}^{s,p}(K)$

Let K be a finite rectilinear cell complex. Recall that N is smoothly embedded in $\mathbb{R}^l$. Let $f, g : |K| \to \mathbb{R}^l$ be two everywhere defined Borel measurable functions. We say that f and g are equivalent if for any $\Delta \in K, f|_\Delta = g|_\Delta$ $\mathcal{H}^d$ a.e. on Δ, where $d = \dim \Delta$. From now on, we identify two such functions and an equivalence class is called a *Borel function.*

Following [40], we introduce the space $\mathcal{W}^{s,p}(K)$ of those Borel functions $f : |K| \to \mathbb{R}^l$ such that for any cell Δ, the restriction $f|_\Delta$ belongs to $W^{s,p}(\Delta)$ and its trace $\mathrm{tr}\, f|_{\partial\Delta}$ is equal to $f|_{\partial\Delta}, \mathcal{H}^{d-1}$*a.e.* $x \in \partial\Delta$.

We write $||f||_{\mathcal{W}^{s,p}(K)} := \sum_{\Delta \in K} ||f|_\Delta||_{W^{s,p}(\Delta)}$.

As in [40], we also define a similar function space as follows. Let K be a finite rectilinear cell complex of dimension m. Assume that

$$|K| = \cup_{\Delta \in K, \dim \Delta = m} \Delta.$$

We define $\tilde{\mathcal{W}}^{s,p}(K)$ as the set of those Borel functions $f : |K| \to \mathbb{R}^l$ such that
i) the map $f|_\Delta \in W^{s,p}(\Delta)$ for any $\Delta \in K$ with $\dim \Delta = m$,
ii) for any $\Sigma \in K$ with $\dim \Sigma = m-1, \Sigma \subset \partial\Delta_i, \dim \Delta_i = m$ for $i = 1, 2$, we have

$$\mathrm{tr}\,(f|_{\Delta_1})|_\Sigma = \mathrm{tr}\,(f|_{\Delta_2})|_\Sigma.$$

We also write:

$$||f||_{\tilde{\mathcal{W}}^{s,p}(K)} = \sum_{\Delta \in K, \dim \Delta = m} ||f|_\Delta||_{W^{s,p}(\Delta)}.$$

Finally, we define

$$\mathcal{W}^{s,p}(K, N) := \{u \in \mathcal{W}^{s,p}(K) : \forall \Delta \in K, u(x) \in N \quad \mathcal{H}^{\dim \Delta} \ a.e.\}$$

and similarly for $\tilde{\mathcal{W}}^{s,p}(K, N)$.

4.6.3 Interpolation

We consider X_0, X_1 two Banach spaces such that X_1 is continuously embedded in X_0. We denote by $||\cdot||_{X_i}$ the norm in $X_i, i = 0, 1$ and for each fixed $t > 0$, we define

$$K(t; u) := \inf\{||u_0||_{X_0} + t||u_1||_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}.$$

Let $1 \leq q < \infty$ and $0 < \theta < 1$. Then we define:

$$(X_0, X_1)_{\theta,q} := \{u \in X_0 : (2^{-i\theta} K(2^i; u))_{i \in \mathbb{Z}} \in l^q(\mathbb{Z})\},$$

which is a Banach space with the norm

$$||u||_{(X_0,X_1)_{\theta,q}} := ||(2^{-i\theta} K(2^i; u))_{i \in \mathbb{Z}}||_{l^q(\mathbb{Z})}.$$

Theorem 4.6 *([1], Theorem 7.48) Let Ω be a rectilinear cell or a smooth bounded open set in $\mathbb{R}^n$. Then we have:*

$$\textit{When} \ \ s \in (0,1), \quad W^{s,p}(\Omega) = (L^p(\Omega), W^{1,p}(\Omega))_{s,p}.$$

$$\textit{When} \ \ s \in (1,2), \quad W^{s,p}(\Omega) = (W^{1,p}(\Omega), W^{2,p}(\Omega))_{s-1,p}.$$

4.6.4 Perturbation

In this section, we follow [40] to explain how we choose *generic* skeletons of a given triangulation of a manifold. Nevertheless, it seems difficult to rewrite exactly the proof of [40] for the case $W^{s,p}$. This is the reason why we use the interpolation method.

Assume that M is an m dimensional Riemannian manifold without boundary. Assume that the parameter space P is a k dimensional Riemannian manifold, Q is a d dimensional Riemannian manifold without boundary, $D \subset Q$ is a domain with compact closure and Lipschitz boundary, and the dimensions satisfy $d + k \geq m$.

In the following, we will need

Lemma 4.12 *Assume $s \in (0,1)$. Let $X_0 := L^p(P, L^p(D))$, $Z_0 := L^p(D)$ and $X_1 := L^p(P,\, W^{1,p}(D))$, $Z_1 := W^{1,p}(D)$. Then we have:*

$$(X_0, X_1)_{s,p} \subset L^p(P, (Z_0, Z_1)_{s,p}) = L^p(P, W^{s,p}(D)).$$

Proof: Let $u \in (X_0, X_1)_{s,p}$ and $\epsilon > 0$. Then, for each $i \in \mathbb{Z}$, there exists $u_0^i \in X_0, u_1^i \in X_1$ such that $u = u_0^i + u_1^i$ and

$$||u_0^i||_{X_0} + 2^i||u_1^i||_{X_1} < K_i(u) + \epsilon/(1+|i|)!$$

where

$$K_i(u) := \inf\{||u_0||_{X_0} + 2^i||u_1||_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}.$$

Then, for $\mathcal{H}^k$ a.e. $\xi \in P, u(\xi) = u_0^i(\xi) + u_1^i(\xi), u_0^i(\xi) \in Z_0, u_1^i(\xi) \in Z_1$. Hence,

$$\inf\{||v_0||_{Z_0} + 2^i||v_1||_{Z_1} : u(\xi) = v_0 + v_1, v_0 \in Z_0, v_1 \in Z_1\} \leq$$
$$||u_0^i(\xi)||_{Z_0} + 2^i||u_1^i(\xi)||_{Z_1}$$

so that

$$||u(\xi)||_{(Z_0,Z_1)_{s,p}} \leq ||(2^{-is}(||u_0^i(\xi)||_{Z_0} + 2^i||u_1^i(\xi)||_{Z_1}))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}.$$

Finally,

$$\begin{aligned}
||u||_{L^p(P,(Z_0,Z_1)_{s,p})} &\leq ||\ ||(2^{-is}(||u_0^i(\cdot)||_{Z_0} + 2^i||u_1^i(\cdot)||_{Z_1}))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}||_{L^p(P)}\\
&= ||(2^{-is}||\ ||u_0^i(\cdot)||_{Z_0} + 2^i||u_1^i(\cdot)||_{Z_1}||_{L^p(P)})_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}\\
&\leq ||(2^{-is}(||u_0^i||_{X_0} + 2^i||u_1^i||_{X_1}))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}\\
&\leq ||(2^{-is}(K_i(u) + \epsilon/(1+|i|!)))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}\\
&\leq ||(2^{-is}K_i(u))_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})} + \epsilon||(2^{-is}/(1+|i|)!)_{i\in\mathbb{Z}}||_{l^p(\mathbb{Z})}\\
&= ||u||_{(X_0,X_1)_{s,p}} + C\epsilon.
\end{aligned}$$

This shows the required inclusion when $\epsilon \to 0$.

$\square$

Similarly, when $s \in (1,2)$, we have:

$$(L^p(P, W^{1,p}(D)), L^p(P, W^{2,p}(D)))_{s-1,p} \subset L^p(P, W^{s,p}(D)). \tag{4.6}$$

Given a map $H : \bar{D} \times P \to M$, we assume that H satisfies:
(H1) $H \in C^2(\bar{D} \times P)$ and $[H(\cdot, \xi)]_{\mathrm{Lip}(\bar{D})} \leq c_0$ for any $\xi \in P$.
(H2) There exists a positive number c_1 such that the m dimensional Jacobian $J_H(x, \xi) \geq c_1, \mathcal{H}^{d+k}$ a.e $(x, \xi) \in \bar{D} \times P$.
(H3) There exists a positive number c_2 such that $\mathcal{H}^{d+k-m}(H^{-1}(y)) \leq c_2$ for $\mathcal{H}^m$ a.e. $y \in M$.

We will denote $H(\cdot, \xi)$ by H_ξ or h_ξ. Then, we have:

Lemma 4.13 *([40], Lemma 3.3) For any Borel function $\chi : M \to \mathbb{R}^+ \cup \{+\infty\}$, we have:*

$$\int_P d\mathcal{H}^k(\xi) \int_D \chi(H_\xi(x))\, d\mathcal{H}^d(x) \leq c_1^{-1} c_2 \int_M \chi(y)\, d\mathcal{H}^m(y).$$

In particular, for any Borel subset $E \subset M$, we have

$$\int_P \mathcal{H}^d(H_\xi^{-1}(E))\, d\mathcal{H}^k(\xi) \leq c_1^{-1} c_2 \mathcal{H}^m(E).$$

If in addition $\mathcal{H}^m(E) = 0$, then $\mathcal{H}^d(H_\xi^{-1}(E)) = 0$ for $\mathcal{H}^k$ a.e. $\xi \in P$.

The following lemma will allow us to give the definition of $[sp]-1$ homotopy.

Lemma 4.14 *i) Let $f \in W^{s,p}(M)$. Then, there exists a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$ and for any $\xi \in P \setminus E, f \circ H_\xi \in W^{s,p}(D)$.*

ii) If we define $\tilde{f}$ by $\tilde{f}(\xi) = f \circ H_\xi$ for any $\xi \in P$, then $\tilde{f} \in L^p(P, W^{s,p}(D))$. In addition,

$$||\tilde{f}||_{L^p(P, W^{s,p}(D))} \leq c||f||_{W^{s,p}(M)},$$

where c depends only on p, c_0, c_1 and c_2.

iii) If $f_i \in C^2(M)$ converges to f in $W^{s,p}(M)$, then $\tilde{f_i}$ converges to $\tilde{f}$ in $L^p(P, W^{s,p}(D))$. Moreover, there exists a subsequence $f_{i'}$ and a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$, and for any $\xi \in P \setminus E, f_{i'} \circ H_\xi \to f \circ H_\xi$ in $W^{s,p}(D)$.

Proof: This lemma corresponds to Lemma 3.4 in [40], the proof of which shows that the map $f \to \tilde{f}$ is continuous from $L^p(M)$ into $L^p(P, L^p(D))$ and from $W^{1,p}(M)$ into $L^p(P, W^{1,p}(D))$. In light of Lemma 4.12, we deduce that this map is continuous from $W^{s,p}(M)$ into $L^p(P, W^{s,p}(D))$ in the case $s \in (0,1)$. This proves ii) when $s \leq 1$. To complete the proof of ii), it remains to consider the case $s \in (1, 1+1/p)$. To this end, we claim that the map $f \to \tilde{f}$ is continuous from $W^{2,p}(M)$ into $L^p(P, W^{2,p}(D))$. This will prove the required result by interpolation as before (using (4.6) instead of Lemma 4.12).

The proof of the claim is similar to the proof of [40] Lemma 3.4., except that $||f||_{W^{1,p}(M)} = ||f||_{L^p(M)} + ||df||_{L^p(M)}$ is replaced by (see [82]):

$$||f||_{W^{2,p}(M)} = ||f||_{L^p(M)} + ||df||_{L^p(M)} + ||d^* df||_{L^p(M)}$$

where d^* is the formal adjoint of the differential operator d on differential forms on M. (The notations $df, d^* df$ have to be understood in a distributional sense).

The rest of the proof is the same and we omit it.

□

Lemma 4.14 implies the following corollary exactly as Lemma 3.4 implies Corollary 3.1 in [40].

Corollary 4.2 *Let $f \in W^{s,p}(M), K$ be a finite rectilinear cell complex, $H : |K| \times P \to M$ be a map such that $H|_{\Delta \times P}$ satisfies $(H1), (H2)$ and $(H3)$ for any $\Delta \in K$. Then, there exists a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$ and for any $\xi \in P \setminus E$, we have $f \circ H_\xi \in \mathcal{W}^{s,p}(K)$; in addition, the map $\tilde{f} = f \circ H_\xi$ for $\xi \in P$ belongs to $L^p(P, \mathcal{W}^{s,p}(K))$.*

4.6.5 Filling a hole (bis)

Lemma 4.3 is valid for any hole diffeomorphic to a ball. When $s \in (1, 1 + 1/p)$, we have a similar result when the 'hole' is a rectilinear cell.

Proposition 4.1 *Let Δ be a rectilinear cell and $y_\Delta \in Int\,\Delta$. Let $u \in W^{s,p}(\Delta)$ be such that $tr u|_{\partial\Delta} = f \in \tilde{\mathcal{W}}^{s,p}(\partial\Delta)$. Then the map u^t defined by*

$$u^t(x) := \begin{cases} u(x/(1-t)) & when\ |x|_\Delta \leq 1 - t, \\ f(x/|x|_\Delta) & when\ |x|_\Delta \geq 1 - t \end{cases}$$

belongs to $C^0([0,1), W^{s,p}(\Delta))$.

Moreover, when $sp < dim\,\Delta$, the map u^t is continuous on $[0,1]$.

We will say that u^1 is the homogeneous degree-zero extension of f.

Proof: We denote by d the dimension of Δ. Let $\Sigma_1, .., \Sigma_r$ be the $d-1$ faces of Δ and $\Delta_1, .., \Delta_r$ be the rectilinear cells defined by

$$\Delta_i := \{\lambda y_\Delta + (1-\lambda)x : x \in \Sigma_i, 0 \leq \lambda \leq 1\}.$$

Since

$$\mathrm{tr}\,(u^t|_{\Delta_i})|_{\Delta_i \cap \Delta_j} = \mathrm{tr}\,(u^t|_{\Delta_j})|_{\Delta_i \cap \Delta_j},$$

in light of Lemma 4.11, it suffices to show that $u^t|_{\Delta_i}$ is continuous into $W^{s,p}(\Delta_i)$.

There exists a C^2 diffeomorphism Φ_i between each Δ_i and a subset of B_1^d of the form $\{\lambda x : \lambda \in [0,1], x \in U_i\}$ where U_i is a connected compact subset of S_1^d, which is isometric in the sense that $|\Phi_i(x)| = |x|_{\Delta_i}, x \in \Delta_i$.

Hence, the continuity of $u^t|_{\Delta_i}$ is a mere consequence of Lemma 4.3. The proposition is proved.

□

4.6.6 The final step for the definition of $[sp] - 1$ homotopy

Let X, Y be topological spaces. Then $[X, Y]$ denotes the set of all homotopy classes of continuous maps from X to Y. Given any $f \in C^0(X, Y)$, we use $[f]_{X,Y}$ (or simply $[f]$) to denote the homotopy class corresponding to f as a map from X to Y. If K is a complex, then for any $f \in \mathcal{W}^{s,p}(K, N)$ and $0 \leq k < sp$, there exists a unique $g \in C^0(K^k, N)$ such that for any $\Delta \in K^k$, we have $f|_\Delta = g|_\Delta$ $\mathcal{H}^d$ a.e. on Δ with $d = \dim \Delta$. Hence, we may define the homotopy class $[f|_{K^k}]$ of f as the homotopy class $[g]$ of g (in $C^0(K^k, N)$).

Lemma 4.15 *(Lemma 4.4 in [40]) Assume that $d \in \mathbb{N}, 1 < d, sp = d, \Delta$ is a rectilinear cell of dimension d and $u \in W^{s,p}(\Delta, N)$ is such that the trace $tr u|_{\partial\Delta} = f \in \tilde{\mathcal{W}}^{s,p}(\partial\Delta, N) \subset C^0(\partial\Delta, N)$. Then, there exists $v \in C^0(\Delta, N) \cap W^{s,p}(\Delta, N)$ such that $v|_{\partial\Delta} = f$ and $v \sim_{W^{s,p}(\Delta,N)} u$.*

Proof: For any $\delta \in (0,1)$, we define $u_\delta(x) = u(x/(1-\delta))$ for $|x|_\Delta \le 1-\delta$ and $u_\delta(x) = f(x/|x|_\Delta)$ for $1-\delta \le |x|_\Delta \le 1$. Then $u_\delta \in W^{s,p}(\Delta)$ and $u_\delta \to u$ in $W^{s,p}(\Delta)$ as $\delta \to 0^+$ (here, we use Proposition 4.1).

Choose an $\eta \in C_c^\infty(\Delta, \mathbb{R})$ such that $0 \le \eta \le 1, \eta|_{\Delta_{1-\delta/2}} = 1$ and $\eta|_{\Delta \setminus \Delta_{1-\delta/3}} = 0$. The notation Δ_r signifies the set $\{x \in \Delta : |x|_\Delta < r\}$. For $\epsilon > 0$ small enough, we set $v_\epsilon(x) = \fint_{B_\epsilon(x)} u_\delta$ for $x \in \Delta_{1-\delta/4}$. Then, we define:

$$w_\epsilon(x) = (1-\eta(x))u_\delta(x) + \eta(x)v_\epsilon(x) \quad \forall x \in \Delta.$$

Clearly, $w_\epsilon \in C^0(\bar{\Delta})$. Since u_δ is VMO, we have $d(v_\epsilon(x), N) \to 0$ uniformly for $x \in \Delta_{1-\delta/2}$, when $\epsilon \to 0^+$ (see [15], section I.2, Example 2). This implies that the same is true for w_ϵ on $\Delta_{1-\delta/2}$ because $v_\epsilon|_{\Delta_{1-\delta/2}} = w_\epsilon|_{\Delta_{1-\delta/2}}$. Moreover, from the uniform continuity of f, we know that $w_\epsilon(x) - u_\delta(x) \to 0$ uniformly for $x \in \Delta \setminus \Delta_{1-\delta/2}$ as $\epsilon \to 0^+$. Hence, $d(w_\epsilon(x), N) \to 0$ uniformly for $x \in \Delta$ as $\epsilon \to 0^+$, from which we deduce that $\Pi_N \circ w_\epsilon$ is well defined for ϵ sufficiently small. We have $v_\epsilon \to u_\delta$ when $\epsilon \to 0^+$ in $W^{s,p}(\Delta)$ (this can be shown as in the case of a regularization by a smooth kernel, see [70], Proposition 4.1.). Then w_ϵ converges to u_δ in $W^{s,p}(\Delta)$ when $\epsilon \to 0^+$. We extend Π_N to the whole $\mathbb{R}^l$ and we may assume that Π_N vanishes outside a large ball. Since Π_N is smooth and N is bounded, by the *composition property* (see [21] and [67]), the map

$$\Pi_N : W^{s,p}(\Delta, N) \to W^{s,p}(\Delta, N)$$

is continuous. Hence $\Pi_N \circ w_\epsilon \to u_\delta$ in $W^{s,p}(\Delta, N)$ when $\epsilon \to 0^+$ and $\Pi_N \circ w_{t\epsilon} \in C^0([0,1], W^{s,p}(\Delta, N))$. Since $u_\delta \sim_{W^{s,p}(\Delta,N)} u$ (by Proposition 4.1), we have $\Pi_N \circ w_\epsilon \sim_{W^{s,p}(\Delta,N)} u$. The map $v := \Pi_N \circ w_\epsilon$ satisfies the requirements of Lemma 4.15.

□

Lemma 4.16 *(Lemma 4.7 in [40]) Let $u \in W^{s,p}(M,N), K$ be a rectilinear cell complex. Assume that the parameter space P is a k dimensional connected Riemannian manifold, and that $H : |K| \times P \to M$ is a map such that $H|_{\Delta \times P}$ satisfies $(H_1), (H_2)$ and (H_3) for any $\Delta \in K$. Then*
i) there exists a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$ and $u \circ H_\xi \in \mathcal{W}^{s,p}(K,N)$ for any $\xi \in P \setminus E$.
ii) Let $0 \le d \le [sp]-1$. We can define $\chi = \chi_{d,H,u} : P \to [|K^d|, N]$ by setting $\chi(\xi) = [u \circ H_\xi|_{|K^d|}]$. Then χ is a constant $\mathcal{H}^k$ a.e. on P.

Proof: From Corollary 4.2 we know that there exists a Borel set $E_0 \subset P$ such that $\mathcal{H}^k(E_0) = 0$ and $u \circ H_\xi \in \mathcal{W}^{s,p}(K, \mathbb{R}^l)$ for any $\xi \in P \setminus E_0$. Since $u(x) \in N$ for almost every $x \in M$, Lemma 4.13 shows that there exists a Borel set $E \subset P$ such that $\mathcal{H}^k(E) = 0$ and $u \circ H_\xi \in \mathcal{W}^{s,p}(K,N)$ for any $\xi \in P \setminus E$; that is, the first assertion of the lemma.

The second assertion can be proved exactly as in [40] Lemma 4.7 except that in the proof, [40] Lemma 4.3 has to be replaced by i) and [40] Lemma 4.4 has to be replaced by our Lemma 4.15.

□

Finally, we give the definition of $[sp]-1$ homotopy (when $s \ge 1$, this definition is the same as in [40]).

Let K be a finite rectilinear cell complex and $h : K \to M$ be a triangulation of M. We define $H : |K| \times B^l_{\epsilon_N} \to M$ as $H(x,\xi) = \Pi_N(h(x)+\xi)$. Then H

satisfies $(H1), (H2)$ and $(H3)$ for each $\Delta \in K$ with $P := B^l_{\epsilon_N}$ (see [40], page 72) so that $\chi_{[sp-1],H,u}$ is a constant a.e. on $B^l_{\epsilon_N}$. We denote this constant by $u_{\sharp,s,p}(h)$. When $s \in (1, 1+1/p), W^{s,p}(M,N) \subset W^{1,sp}(M,N)$ (because N is a bounded subset of $\mathbb{R}^l$) and $u_{\sharp,s,p}(h)$ is exactly the constant $u_{\sharp,sp}(h)$ defined in [40] (for $s = 1$).

We also remark that for ϵ_N sufficiently small, $H(\cdot, \xi)$ is a triangulation of M (see [73]). We will denote $H(\cdot, \xi)$ by H_ξ or h_ξ.

Lemma 4.8 and Lemma 4.9 in [40] show that if $u, v \in W^{s,p}(M,N)$ and $h_i : K_i \to M$ are triangulations for $i = 1, 2$ (K_i being a rectilinear cell complex) and $u_{\sharp,s,p}(h_1) = v_{\sharp,s,p}(h_1)$, then $u_{\sharp,s,p}(h_2) = v_{\sharp,s,p}(h_2)$. In fact, when $s \in (0,1)$, the same proof as in the case $s = 1$ is valid. When $s \in (1, 1+1/p)$, one can use the inclusion $W^{s,p}(M,N) \subset W^{1,sp}(M,N)$ and apply directly the results in [40] with sp instead of p. Hence, we can define:

Definition 4.1 *Let $u, v \in W^{s,p}(M,N)$. If for any Lipschitz rectilinear cell decomposition $h : K \to M$, we have $u_{\sharp,s,p}(h) = v_{\sharp,s,p}(h)$, then we say that u is $[sp]-1$ homotopic to v.*

Clearly, this is an equivalence relation on $W^{s,p}(M,N)$.

4.7 A preliminary to the proof of Theorem 4.4

In [40], the fact that $\mathrm{Lip}(\Delta) \subset W^{1,p}(\Delta)$ for any simplex Δ is widely used. In contrast, $\mathrm{Lip}(\Delta) \not\subset W^{s,p}(\Delta)$ when $s > 1$. To overcome this difficulty, we have to substantially modify some parts of the proofs of [40]. This is the aim of this section.

Throughout this section, X denotes a rectilinear cell complex of dimension $k+1$ with $0 \le k \le sp-1$ and X^k its subcomplex of dimension k. We also define $[0,1] \times X^k \cup \{0\} \times X$ as the complex:

$$\{[0,1] \times \Delta : \Delta \in X^k\} \cup \{\{0\} \times \Delta : \Delta \in X\} \cup \{\{1\} \times \Delta : \Delta \in X^k\}.$$

If X is embedded in some $\mathbb{R}^S$ and $\Delta \in X^k$, then $[0,1] \times \Delta$ is a rectilinear cell in $\mathbb{R} \times \mathbb{R}^S$ and its boundary is

$$\{0\} \times \Delta \cup \{1\} \times \Delta \cup [0,1] \times \partial\Delta \ \subset \ [0,1] \times X^k \cup \{0\} \times X.$$

The proof of [40], Lemma 3.2 (with obvious modifications) shows the following

Lemma 4.17 *The set $C^0(X) \cap \mathcal{W}^{s,p}(X)$ is dense in the set $C^0(X)$.*

A consequence of Lemma 4.17 is given by

Lemma 4.18 *Let $H_0 \in C^0([0,1] \times X^k, N)$ be such that $H_0(0,\cdot)$ and $H_0(1,\cdot)$ belong to $\mathcal{W}^{s,p}(X^k, N)$. Then there exists*

$$H_1 \in \mathcal{W}^{s,p}([0,1] \times X^k, N) \cap C^0([0,1] \times X^k, N)$$

such that $H_0(0,\cdot) = H_1(0,\cdot)$ and $H_0(1,\cdot) = H_1(1,\cdot)$.

Proof: First, we may assume that $H_0(t,\cdot) = H_0(0,\cdot)$, $t \in [0,\delta]$ and $H_0(t,\cdot) = H_0(1,\cdot)$, $t \in [1-\delta,1]$, for some $\delta \in (0,1/4)$. Moreover, using Lemma 4.17, there exists G in $\mathcal{W}^{s,p}([0,1]\times X^k)\cap C^0([0,1]\times X^k)$ such that $|G(t,x)-H_0(t,x)| \le \epsilon_N$ for $(t,x) \in [0,1]\times |X^k|$.

Finally, let $\theta \in C^\infty(\mathbb{R},[0,1])$ such that $\theta \equiv 1$ on $[\delta/2, 1-\delta/2]$ and $\theta \equiv 0$ on $[0,\delta/4]\cup[1-\delta/4,1]$. Then we define

$$H(t,x) := \theta(t)G(t,x) + (1-\theta(t))H_0(t,x).$$

The map H belongs to

$$H \in \mathcal{W}^{s,p}([0,1]\times X^k,\mathbb{R}^l)\cap C^0([0,1]\times X^k,\mathbb{R}^l)$$

and $|H(t,x)-H_0(t,x)| \le \epsilon_N$.

Thus, we can define $H_1(t,x) := \Pi_N \circ H(t,x)$. By the composition property, $H_1 \in \mathcal{W}^{s,p}([0,1]\times X^k,N)\cap C^0([0,1]\times X^k,N)$. We have $H_1(0,\cdot) = H_0(0,\cdot)$ and $H_1(1,\cdot) = H_0(1,\cdot)$. This completes the proof of the lemma.

□

Lemma 4.19 *Let $H_1 \in \mathcal{W}^{s,p}([0,1]\times X^k\cup\{0\}\times X,N)\cap C^0([0,1]\times X^k\cup\{0\}\times X,N)$. Then H_1 may be extended to a map*

$$H_2 \in \mathcal{W}^{s,p}([0,1]\times X,N)\cap C^0([0,1]\times X,N).$$

Proof: For each $\Delta \in X\setminus X^k$, consider its barycenter y_Δ and define $\bar y_\Delta := (2,y_\Delta) \in \bar\Delta := [0,4]\times\Delta$. Let ρ be the map defined on $[0,1]\times\Delta$ by

$$x \mapsto \bar y_\Delta + (x-\bar y_\Delta)/|x|_{\bar\Delta}.$$

Then

$$\rho(x) \in [0,1]\times\partial\Delta\cup\{0\}\times\Delta, \quad x\in[0,1]\times\Delta$$

and $\rho(x) = x$ for any $x \in [0,1]\times\partial\Delta\cup\{0\}\times\Delta$. Define ρ on each such $[0,1]\times\Delta$ for $\Delta \in X\setminus X^k$ and extend it to $[0,1]\times|X|$ by setting $\rho(x) = x$ on $[0,1]\times|X^k|$. Then ρ is a Lipschitz map from $[0,1]\times|X|$ into $[0,1]\times|X^k|\cup\{0\}\times|X|$, so that the map $H_2 := H_1\circ\rho$ belongs to $C^0([0,1]\times X,N)$. Moreover, H_2 is an extension of H_1. To see that $H_2 \in \mathcal{W}^{s,p}([0,1]\times X,N)$, remark that on each cell $[0,1]\times\Delta$, with $\Delta\in X\setminus X^k$, H_2 is defined as the homogeneous degree-zero extension of H_1 (except that the center of the homogeneous degree-zero extension $\bar y_\Delta$ does not belong to the cell, which makes no trouble as the proof of Proposition 4.1 shows). Hence, $H_2|_{[0,1]\times\Delta} \in W^{s,p}$. That $H_2|_{\{1\}\times\Delta}\in W^{s,p}$ is an easy consequence of the fact that $H_1 \in \mathcal{W}^{s,p}([0,1]\times\partial\Delta\cup\{0\}\times\Delta)$ and that ρ^{-1} defined on the complex $[0,1]\times\partial\Delta\cup\{0\}\times\Delta$ is a triangulation of $\{1\}\times\Delta$ (see the remarks before Lemma 4.11). The lemma is proved.

□

Lemma 4.20 *Let $H_2 \in C^0([0,1]\times X,N)$ be such that $H_2(0,\cdot)$ and $H_2(1,\cdot)$ belong to $\mathcal{W}^{s,p}(X,N)$. Then there exists $H_3 \in C^0([0,1],\mathcal{W}^{s,p}(X,N))$ such that $H_3(0) = H_2(0,\cdot)$ and $H_3(1) = H_2(1,\cdot)$.*

Proof: There exists $\delta > 0$ such that $|H_2(t_1,x_1) - H_2(t_2,x_2)| \le \epsilon_N/8$ for any $|x_1-x_2| + |t_1-t_2| \le \delta$. Pick some $m\in\mathbb{N}$ such that $1/m < \delta$. For any $1 \le k \le$

$m-1$, there exists $L_{k/m} \in C^0(X) \cap \mathcal{W}^{s,p}(X)$ such that $|L_{k/m}(x) - H_2(k/m, x)| \leq \epsilon_N/8$ for $x \in |X|$. (Here, we use Lemma 4.17). We also define $L_0 := H_2(0, \cdot)$ and $L_1 := H_2(1, \cdot)$. For any $0 \leq k \leq m-1$, $t \in [k/m, (k+1)/m]$ and $x \in X$, we define

$$L(t)(x) = (k+1-mt)L_{k/m}(x) + (mt-k)L_{(k+1)/m}(x).$$

It is easy to see that

$$L \in C^0([0,1], \mathcal{W}^{s,p}(X, \mathbb{R}^l)) \cap C^0([0,1] \times X, \mathbb{R}^l)$$

and $\operatorname{dist}(L(t)(x), N) < \epsilon_N, t \in [0,1], x \in |X|$.

We define $H_3(t)(x) := \Pi_N(L(t)(x))$. The composition property shows that the map $t \in [0,1] \mapsto \Pi_N \circ L(t) \in W^{s,p}(\Delta, N)$ is continuous for each $\Delta \in X$. This implies that $H_3 \in C^0([0,1], \mathcal{W}^{s,p}(X, N))$.

□

The proof of Theorem 4.4 is mainly based on the following proposition:

Proposition 4.2 *Let $u, v \in \mathcal{W}^{s,p}(X, N)$. Then $u|_{|X^k|}$ and $v|_{|X^k|}$ can be identified to elements in $C^0(X^k, N)$. Assume that $u|_{|X^k|} \sim_{C^0(X^k,N)} v|_{|X^k|}$. Then there exists $f \in \mathcal{W}^{s,p}(X, N) \cap C^0(X, N)$ such that $u \sim_{\mathcal{W}^{s,p}(X,N)} f$ and $f|_{|X^k|} = v|_{|X^k|}$.*

Proof: First, we claim that we may assume that $u \in C^0(X, N)$. Indeed, if $sp > k+1$, then this is a consequence of Sobolev's embeddings. If $sp = k+1$, then Lemma 4.15 applied to each $\Delta \in X \setminus X^k$ shows that there exists $u_1 \in \mathcal{W}^{s,p}(X, N) \cap C^0(X, N)$ such that $u_1|_{|X^k|} = u|_{|X^k|}$ and $u_1 \sim_{\mathcal{W}^{s,p}(X,N)} u$.

There exists $H_0 \in C^0([0,1] \times X^k, N)$ such that $H_0(0, \cdot) = u|_{|X^k|}$ and $H_0(1, \cdot) = v|_{|X^k|}$. Using Lemma 4.18, there exists

$$H_1 \in \mathcal{W}^{s,p}([0,1] \times X^k, N) \cap C^0([0,1] \times X^k, N)$$

such that $H_1(0, \cdot) = H_0(0, \cdot)$ and $H_1(1, \cdot) = H_0(1, \cdot)$.

Then extend H_1 to a map still denoted by H_1, defined on $[0,1] \times X^k \cup \{0\} \times X$ by setting $H_1(0, x) = u(x)$ for $x \in X$. It is clear that H_1 now belongs to the space

$$\mathcal{W}^{s,p}([0,1] \times X^k \cup \{0\} \times X, N) \cap C^0([0,1] \times X^k \cup \{0\} \times X, N).$$

In light of Lemma 4.19, we may extend H_1 to a map

$$H_2 \in \mathcal{W}^{s,p}([0,1] \times X, N) \cap C^0([0,1] \times X, N).$$

Finally, using Lemma 4.20, there exists $H_3 \in C^0([0,1], \mathcal{W}^{s,p}(X, N))$ such that $H_3(0) = H_2(0, \cdot) = u$ and $H_3(1) = H_2(1, \cdot)$. We have $H_2(1, \cdot)|_{|X^k|} = v|_{|X^k|}$. We can set $f := H_3(1)$.

□

4.8 Proof of Theorem 4.4

Lemma 4.21 *There exists $\eta > 0$ such that for any $u, v \in W^{s,p}(M, N)$ satisfying $||u - v||_{W^{s,p}(M, \mathbb{R}^l)} < \eta$, we have*

$$u \text{ is } [sp] - 1 \text{ homotopic to } v.$$

Proof: Fix a smooth triangulation of M, say $h : K \to M$. We may find a Borel set $E_1 \subset B^l_{\epsilon_N}$ such that $\mathcal{H}^l(E_1) = 0$ and for any $\xi \in B^l_{\epsilon_N} \setminus E_1$, we have $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{s,p}(K, N)$ and

$$[u \circ h_\xi|_{|K^{[sp]-1}|}] = u_{\sharp,s,p}(h) \ , \ [v \circ h_\xi|_{|K^{[sp]-1}|}] = v_{\sharp,s,p}(h).$$

For any $\Delta \in K$, we have (see Lemma 4.14)

$$\int_{B^l_{\epsilon_N}} d\mathcal{H}^l(\xi)||u \circ h_\xi - v \circ h_\xi||^p_{W^{s,p}(\Delta,\mathbb{R}^l)} \leq C||u - v||^p_{W^{s,p}(M,\mathbb{R}^l)}.$$

This implies:

$$\mathcal{H}^l(\{\xi \in B^l_{\epsilon_N} : ||u \circ h_\xi - v \circ h_\xi||^p_{W^{s,p}(\Delta,\mathbb{R}^l)} \geq r\}) \leq C\frac{\epsilon^l_N||u - v||^p_{W^{s,p}(M,\mathbb{R}^l)}}{r}.$$

Hence, we may find a Borel set $E_2 \subset B^l_{\epsilon_N}$ such that $\mathcal{H}^l(E_2) > 0$ and for any $\xi \in E_2$, we have:
(i) $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{s,p}(K, N)$
(ii) For any $\Delta \in K$, we have

$$||u \circ h_\xi - v \circ h_\xi||^p_{W^{s,p}(\Delta,\mathbb{R}^l)} \leq C||u - v||^p_{W^{s,p}(M,\mathbb{R}^l)}.$$

Hence, for any $\Delta \in K^{[sp-1]}$, we have:

$$\begin{aligned} ||u \circ h_\xi - v \circ h_\xi||_{L^\infty(\Delta)} &\leq C||u \circ h_\xi - v \circ h_\xi||_{W^{s,p}(\Delta,\mathbb{R}^l)} \\ &\leq C||u - v||_{W^{s,p}(M,\mathbb{R}^l)}. \end{aligned}$$

If $||u - v||_{W^{s,p}(M,\mathbb{R}^l)} \leq \eta := \epsilon_N/C$, then the continuous map

$$H(t, x) := \Pi_N((1 - t)u \circ h_\xi(x) + tv \circ h_\xi(x))$$

is well defined. This shows that u is $[sp] - 1$ homotopic to v.

□

Lemma 4.21 will allow us to prove one implication of Theorem 4.2. For the converse of this implication, we will need the two following propositions.

Proposition 4.3 *Assume that $1 < sp < d$ and that f is a continuous path in $\tilde{\mathcal{W}}^{s,p}(\partial\Delta, N)$, where Δ is a d dimensional rectilinear cell containing 0. Define $\tilde{f}(t)(x) = f(t)(x/|x|)$ for $0 \leq t \leq 1$ and $x \in \Delta$. (Here, $|\cdot|$ denotes the Minkowski functional of Δ with respect to 0). Then $\tilde{f}$ is a continuous path in $W^{s,p}(\Delta, N)$.*

Proof: In light of the proof of Proposition 4.1, Lemma 4.1 and (4.5), the proposition follows from

$$||\tilde{f}(t) - \tilde{f}(s)||_{W^{s,p}(\Delta)} = ||\widetilde{f(t) - f(s)}||_{W^{s,p}(\Delta)} \leq C||f(t) - f(s)||_{\tilde{\mathcal{W}}^{s,p}(\partial\Delta)}.$$

□

Proposition 4.4 *Consider a d dimensional rectilinear cell Δ containing 0. Assume that $1 < sp < d$. Let $u, v \in W^{s,p}(\Delta, N)$ be such that $tru|_{\partial\Delta}, trv|_{\partial\Delta} \in \tilde{\mathcal{W}}^{s,p}(\partial\Delta, N)$ and $tru|_{\partial\Delta} \sim_{\tilde{\mathcal{W}}^{s,p}(\partial\Delta,N)} trv|_{\partial\Delta}$. Then $u \sim_{W^{s,p}(\Delta,N)} v$.*

Proof: There exists $f \in C^0([0,1], \tilde{\mathcal{W}}^{s,p}(\partial\Delta, N))$ such that $\operatorname{tr} u = f(0)$, $\operatorname{tr} v = f(1)$. Then, Proposition 4.3 implies the existence of some

$$\tilde{f} \in C^0([0,1], W^{s,p}(\Delta, N))$$

satisfying $\tilde{f}(0) = \tilde{u}, \tilde{f}(1) = \tilde{v}$ with $\tilde{u}(x) = \operatorname{tr} u|_{\partial\Delta}(x/|x|)$ and similarly for $\tilde{v}$. Moreover, Proposition 4.1 shows that $\tilde{u} \sim_{W^{s,p}(\Delta)} u$, $\tilde{v} \sim_{W^{s,p}(\Delta)} v$. Finally, $u \sim_{W^{s,p}(\Delta)} v$.

□

We proceed to prove Theorem 4.4; that is,

Theorem 4.7 *Let $u, v \in W^{s,p}(M,N)$. Then $u \sim_{s,p} v$ if and only if u is $[sp]-1$ homotopic to v in $W^{s,p}(M,N)$.*

Proof: Let $u, v \in W^{s,p}(M,N)$. Assume that $u \sim_{s,p} v$. Then there exists a continuous map $H \in C^0([0,1], W^{s,p}(M,N))$ such that $H(0,\cdot) = u$ and $H(1,\cdot) = v$.

Let η be the number in Lemma 4.21. There exists $m \in \mathbb{N}$ such that for any $s, t \in [0,1]$ satisfying $|s-t| \leq 1/m$, we have:

$$||H(s) - H(t)||_{W^{s,p}(M,\mathbb{R}^l)} < \eta.$$

Then, for $i = 0, .., m-1$, we have $H(i/m)$ is $[sp]-1$ homotopic to $H((i+1)/m)$. This proves that u is $[sp]-1$ homotopic to v.

The converse is very close to [40]. Suppose that we are given two maps $u, v \in W^{s,p}(M,N)$ which are $[sp]-1$ homotopic. For convenience, we note $k = [sp]-1$. Let $h : K \to M$ be a smooth triangulation of M.

By definition of $[sp]-1$ homotopy, we may find a $\xi \in B^l_{\epsilon_0}$ such that $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{s,p}(K,N)$ and $u \circ h_\xi|_{|K^k|} \sim v \circ h_\xi|_{|K^k|}$ as maps from $|K^k|$ to N. We remark that it is enough to prove that $u \circ h_\xi$ and $v \circ h_\xi$ are $\tilde{W}^{s,p}(K,N)$ homotopic. Indeed, if this is the case, u and v will be $W^{s,p}(h_\xi(\Delta), N)$ homotopic for each $\Delta \in K$ of dimension m (recall that h_ξ is a smooth diffeomorphism from Δ onto $h_\xi(\Delta)$). Then, Lemma 4.11 implies that $u \sim_{W^{s,p}(M,N)} v$.

Step 1: a reduction. We claim that we can assume that $u \circ h_\xi|_{|K^k|} = v \circ h_\xi|_{|K^k|}$. Indeed, since $u \circ h_\xi|_{|K^k|} \sim v \circ h_\xi|_{|K^k|}$ as maps from $|K^k|$ to N, we may apply Proposition 4.2 which shows that $u \circ h_\xi|_{K^{k+1}}$ is $\mathcal{W}^{s,p}(K^{k+1}, N)$ homotopic to a map $f \in \mathcal{W}^{s,p}(K^{k+1}, N) \cap C^0(K^{k+1}, N)$ which coincides with v on $|K^k|$. For each $(k+2)$ simplex Δ, f and $\operatorname{tr} u \circ h_\xi|_{\partial\Delta} = u \circ h_\xi|_{\partial\Delta}$ belongs to $\mathcal{W}^{s,p}(\partial\Delta)$. We choose the barycenter of Δ as origin and do homogeneous degree-zero extension from f to get $f_\Delta \in W^{s,p}(\Delta, N)$ on Δ. Define f_Δ on each such Δ to get $f_{k+2} \in \mathcal{W}^{s,p}(K^{k+2}, N)$. Proposition 4.4 shows that $u \circ h_\xi|_{K^{k+2}}$ is homotopic to f_{k+2} in $\mathcal{W}^{s,p}(K^{k+2}, N)$. Simply by induction we finish after working with n simplices.

Then, $u \circ h_\xi$ is $\mathcal{W}^{s,p}(K,N)$ homotopic to f. This completes the proof of step 1.

Step 2: completion of the proof. We now show that f can be connected to $v \circ h_\xi$ by a continuous path in $\tilde{\mathcal{W}}^{s,p}(K,N)$.

Applying Proposition 4.1 to each $k+1$ simplex $\Delta \in K$, we may assume that $f|_{\Delta \setminus B_\delta(c_\Delta)} = v \circ h_\xi|_{\Delta \setminus B_\delta(c_\Delta)}$. Here c_Δ is the barycenter of Δ and δ is a small number. Note that f is continuous on Δ and that v is continuous on $\Delta \setminus B_\delta(c_\Delta)$.

Doing homogeneous degree-zero extension from $v \circ h_\xi|_{K^{k+1}}$ and $f|_{K^{k+1}}$ as we have done above, we may assume that $v \circ h_\xi$ and f are homogeneous of degree zero on $\Sigma \in K$ with $\dim \Sigma \geq k+2$. Then, on any $k+2$ simplex $\Sigma \in K$, f is continuous on $\Sigma \setminus \{c_\Sigma\}$ and $v \circ h_\xi$ is continuous on $\Sigma \setminus \{tz + (1-t)c_\Sigma : z \in \bar{B}_\delta(c_\Delta), t \in [0,1]\}$ (here, c_Σ is the barycenter of Σ and the center of the homogeneous degree-zero extension on Σ).

Fix a $k+1$ simplex Δ. It must be the face of several $k+2$ simplices, say $\Sigma_1, .., \Sigma_r, r \geq 2$. Now, for two small numbers $\delta' > \delta$ and $\epsilon > 0$, consider $\Omega := \cup_{i=1}^r \Omega_i$ where $\Omega_i \subset \Sigma_i$ is formally equal to $(\bar{B}_{2\delta'}(c_\Delta) \cap \Delta) \times [0, \epsilon]$, for which the product means that we go in the Σ_i in the normal direction by length ϵ. Define

$$\Omega_i' := (\bar{B}_{2\delta'}(c_\Delta) \cap \Delta) \times [0, \frac{1}{2}\epsilon], \Omega_i'' := (\bar{B}_{2\delta'}(c_\Delta) \cap \Delta) \times [\epsilon/2, \epsilon],$$

$$\Omega' = \cup_{i=1}^r \Omega_i', \Omega'' = \cup_{i=1}^r \Omega_i''.$$

We may choose δ' and ϵ such that $f|_{\partial\Omega_i \cup \partial\Omega_i''} \in \tilde{\mathcal{W}}^{s,p}(\partial\Omega_i \cup \partial\Omega_i'')$ and $v \circ h_\xi \in \tilde{\mathcal{W}}^{s,p}(\partial\Omega_i')$ (this amounts to Lemma 4.2 i); note also that the trace compatibility conditions are automatically satisfied for $\delta' > \delta$ and $\epsilon > 0$ sufficiently small: this follows from the continuity properties of f and $v \circ h_\xi$ stated above). This implies that $f|_{\partial\Omega} \in \tilde{\mathcal{W}}^{s,p}(\partial\Omega)$ (once again, the trace compatibility conditions are satisfied). If ϵ is taken sufficiently small (this depends only on the geometry of the $k+2$ simplices), we can assume that $v \circ h_\xi = f$ on a neighborhood of $\partial\Omega' \cap \partial\Omega$ (recall that on K^{k+2}, f and $v \circ h_\xi$ are now homogeneous of degree zero).

Now consider a w defined on $|K^{k+2}|$ by setting $w|_{\Omega'} = v \circ h_\xi, w|_{|K^{k+2}|\setminus\Omega} = f|_{|K^{k+2}|\setminus\Omega}$. On each Ω_i'', we simply do homogeneous degree-zero extension with respect to a point in $\operatorname{int}\Omega_i''$ (here, we use the fact that the map equal to f on $\partial\Omega_i'' \setminus \partial\Omega_i'$ and equal to $v \circ h_\xi$ on $\partial\Omega_i'' \cap \partial\Omega_i' = (\bar{B}_{2\delta}(c_\Delta) \cap \Delta) \times \{\epsilon/2\}$ belongs to $\tilde{\mathcal{W}}^{s,p}(\partial\Omega_i'')$). Clearly, $w \in \tilde{\mathcal{W}}^{s,p}(K^{k+2}, N)$.

We may connect w to $f|_{|K^{k+2}|}$ by a continuous path in $\tilde{\mathcal{W}}^{s,p}(K^{k+2}, N)$ since for any $1 \leq i \neq j \leq r$, $\Omega_i \cup \Omega_j$ is star-shaped with respect to c_Δ and we may apply Proposition 4.1 to w on this set (here, we use the fact that $w|_{\partial(\Omega_i \cup \Omega_j)} = f|_{\partial(\Omega_i \cup \Omega_j)}$ belongs to $\tilde{\mathcal{W}}^{s,p}(\partial(\Omega_i \cup \Omega_j))$).

Define $\tilde{w}$ inductively to be the homogeneous degree-zero extension of w on each higher-dimensional simplex Δ with $\dim \Delta \geq k+3$, from its value on $\partial\Delta$ as described above. Then, one has $\tilde{w} \sim_{\tilde{\mathcal{W}}^{s,p}(K,N)} f$.

Since $\tilde{w}|_{|K^{k+1}|} = v \circ h_\xi|_{|K^{k+1}|}$, we have $\tilde{w} \sim_{\tilde{\mathcal{W}}^{s,p}(K,N)} v \circ h_\xi$ (by Proposition 4.4 and Lemma 4.11). Finally, $v \circ h_\xi \sim_{\tilde{\mathcal{W}}^{s,p}(K,N)} u \circ h_\xi$. This completes the proof of the theorem.

□

4.9 Consequences of Theorem 4.4

As in [40], Theorem 4.4 reduces certain problems about Sobolev mappings, which are analytical problems, to pure topology problems. In this section, we enumerate some of these results, which correspond to similar results in [40] (for $W^{1,p}$). We omit their proofs when they are similar to those of [40].

Proposition 4.5 *([40], Proposition 5.1) Assume that* $1 \leq p$, $s \in (0, 1+1/p)$, $1 < sp < m$. *For any triangulation of* M, *say* $h : K \to M$, *we set* $M^j = h(|K^j|)$ *for any* j. *There is a bijection between the sets* $W^{s,p}(M,N)/\sim_{s,p}$ *and* $C^0(M^{[sp]}, N)/\sim_{M^{[sp]-1}}$. *Here for* $f, g \in C^0(M^{[sp]}, N)$, $f \sim_{M^{[sp]-1}} g$ *means that* $f|_{M^{[sp]-1}}$ *and* $g|_{M^{[sp]-1}}$ *are homotopic in* $C^0(M^{[sp]-1}, N)$.

Proof: A way to show this proposition is to introduce the space

$$X := (C^0(M^{[sp]}, N) \cap \mathcal{W}^{s,p}(M^{[sp]}, N))/\sim_{M^{[sp]-1}} .$$

The definition of $\mathcal{W}^{s,p}(M^{[sp]}, N)$ follows exactly the definition of $\mathcal{W}^{s,p}(K, N)$.

The natural map $G : X \to C^0(M^{[sp]}, N)/\sim_{M^{[sp]-1}}$ is one-to-one. The surjectivity of G is an easy consequence of Lemma 4.17. Indeed, let $u \in C^0(M^{[sp]}, N)$. Then Lemma 4.17 shows that there exists $v \in C^0(M^{[sp]}) \cap \mathcal{W}^{s,p}(M^{[sp]})$ such that $||u - v||_{L^\infty(M^{[sp]})} < \epsilon_N$ and $||\Pi_N(v) - u||_{L^\infty(M^{[sp]})} < \epsilon_N$. Hence u is continuously connected to $\Pi_N(v) \in C^0(M^{[sp]}, N) \cap \mathcal{W}^{s,p}(M^{[sp]}, N)$ by the map $H(t) := \Pi_N(t\Pi_N(v) + (1-t)u)$, so that $G(\Pi_N(v)) = u$.

Thus, there is a bijection between $C^0(M^{[sp]}, N)/\sim_{M^{[sp]-1}}$ and X. It remains to show that there is a bijection between X and $W^{s,p}(M,N)/\sim_{s,p}$.

We define a map from X into $W^{s,p}(M,N)/\sim_{s,p}$ as follows: For any $w \in C^0(M^{[sp]}, N) \cap \mathcal{W}^{s,p}(M^{[sp]}, N)$, using h to pull w to $K^{[sp]}$, after doing homogeneous degree-zero extension on higher-dimensional cells, we pull it to M by h and get $\tilde{w}$. Then we send the equivalence class corresponding to w to the equivalence class corresponding to $\tilde{w}$. This map is well defined by the proof of Theorem 4.4.

We proceed to prove that this map is one-to-one. Let $u, v \in C^0(M^{[sp]}, N) \cap \mathcal{W}^{s,p}(M^{[sp]}, N)$ and $\tilde{u}, \tilde{v}$ their homogeneous degree-zero extension. Assume that $\tilde{u} \sim_{s,p} \tilde{v}$. Then by Theorem 4.4, $\tilde{u}_{\sharp,s,p}(h) = \tilde{v}_{\sharp,s,p}(h)$. It is easy to see that $\tilde{u}_{\sharp,s,p}(h) = [u \circ h|_{K^{[sp]-1}}]$ and similarly for v. Hence $u \sim_{M^{[sp]-1}} v$; that is, the map is one-to-one.

To prove the surjectivity, let $u \in W^{s,p}(M,N)$. There exists $\xi \in B^l_{\epsilon_N}$ such that $u \circ h_\xi \in \mathcal{W}^{s,p}(K, N)$. By the Sobolev embeddings or Lemma 4.15, there exists $f \in C^0(K^{sp}, N) \cap \mathcal{W}^{s,p}(K^{[sp]}, N)$ such that $f|_{|K^{[sp]-1}|} = u \circ h_\xi|_{|K^{[sp]-1}|}$. We extend f by degree-zero homogeneity. We denote by $\tilde{f}$ this extension. The proof of Theorem 4.4 (in fact, this is exactly 'step 2') shows that $u \circ h_\xi \sim_{\tilde{\mathcal{W}}^{s,p}(K,N)} \tilde{f}$. Hence, $u \circ h_\xi \circ h^{-1} \sim_{W^{s,p}(M,N)} \tilde{f} \circ h^{-1}$. Since $u \circ h_\xi \circ h^{-1} \sim_{W^{s,p}(M,N)} u$, the equivalence class corresponding to $f \circ h^{-1}|_{M^{[sp]}}$ is mapped to the equivalence class corresponding to u. That is, the map is onto.

□

For any $0 < s_1, s_2 \leq 1, 1 \leq p_1, p_2$, such that $W^{s_2,p_2} \subset W^{s_1,p_1}$, we have a map:

$$i : W^{s_2,p_2}/\sim_{s_2,p_2} \to W^{s_1,p_1}/\sim_{s_1,p_1}$$

defined in an obvious way. An immediate consequence of the above proposition is the following

Corollary 4.3 *([40], Corollary 5.1) Assume that* $[s_1p_1] = [s_2p_2]$. *Then* i *is a bijection.*

The following corollary implies Theorem 4.3 b).

Corollary 4.4 *([40], Corollary 5.2) Assume that* $1 \leq p, s \in (0, 1+1/p), 1 < sp < \dim M$, *and* $\pi_i(N) = 0$ *for* $[sp] \leq i \leq \dim M$. *Then there is a bijection between* $C^0(M,N)/\sim$ *and* $W^{s,p}(M,N)/\sim_{s,p}$.

Corollary 4.5 *([40], Corollary 5.3) Assume that* $1 \leq p, s \in (0, 1+1/p), 1 < sp < m$. *If there exists some* $k \in \mathbb{Z}, k \leq [sp]-1$ *such that* $\pi_i(M) = 0$ *for* $1 \leq i \leq k$, *and* $\pi_i(N) = 0$ *for* $k+1 \leq i \leq [sp]-1$, *then* $W^{s,p}(M,N)$ *is path-connected.*

This is Theorem 4.2.

We now turn to the question whether a given Sobolev map in $W^{s,p}(M,N)$ can be connected to a smooth map by a continuous path in $W^{s,p}(M,N)$. It turns out that there is a necessary and sufficient topological condition for this to be true.

Proposition 4.6 *([40], Proposition 5.2) Assume that* $1 \leq p, s \in (0, 1+1/p)$, $1 < sp < m$, $u \in W^{s,p}(M,N)$, *and that* $h : K \to M$ *is a triangulation. Then,* u *can be connected to a smooth map by a continuous path in* $W^{s,p}(M,N)$ *if and only if* $u_{\sharp,s,p}(h)$ *is extendible to* M *with respect to* N, *that is: for any* $f \in C^0(K^{[sp]-1}, N)$ *such that* $f \in u_{\sharp,s,p}(h), f$ *is the restriction of a map in* $C^0(K,N)$.

Corollary 4.6 *([40], Corollary 5.4) Assume that* $1 \leq p, s \in (0, 1+1/p), 1 < sp < m$. *Then every map in* $W^{s,p}(M,N)$ *can be connected by a continuous path in* $W^{s,p}(M,N)$ *to a smooth map if and only if* M *satisfies the* $[sp]-1$ *extension property with respect to* N, *that is: there exists a CW complex structure* $(M^j)_{j\in\mathbb{Z}}$ *of* M *such that every* $f \in C^0(M^{[sp]}, N), f|_{M^{[sp]-1}}$ *has a continuous extension to* M.

This is Theorem 4.5 e).

Proof: Fix a smooth triangulation of M, say $h : K \to M$. Assume that every map in $W^{s,p}(M,N)$ can be connected continuously to a smooth map. Let $f \in C^0(M^{[sp]}, N)$. Then using Lemma 4.17, there exists $f_1 \in C^0(K^{[sp]}, N) \cap \mathcal{W}^{s,p}(K^{[sp]}, N)$ such that $f_1 \sim_{C^0(K^{[sp]},N)} f \circ h$. Let g be the homogeneous degree-zero extension of f_1 to K. Then $u = g \circ h^{-1} \in W^{s,p}(M,N)$ and $u_{\sharp,s,p}(h) = [g|_{K^{[sp]-1}}]$. Since u can be connected continuously to a smooth map, from Proposition 4.6 we know that $f_1|_{|K^{[sp]-1}|}$ has a continuous extension to K with respect to N. Hence, $f|_{M^{[sp]-1}}$ has a continuous extension to M.

Conversely, assume that M satisfies the $([sp]-1)$ extension property with respect to N. Given any $u \in W^{s,p}(M,N)$, there exists $\xi \in B^l_{\epsilon_N}$ such that $u \circ h_\xi \in \mathcal{W}^{s,p}(K,N)$ and $u_{\sharp,s,p}(h) = [u \circ h_\xi|_{|K^{[sp]-1}|}]$. Using the Sobolev embeddings or Lemma 4.15, we may assume that $u \circ h_\xi \in C^0(K^{[sp]}, N)$. Hence, by Proposition 4.6, u may be connected continuously to a smooth map.

$\square$

Chapter 5

Topological singularities in $W^{s,p}(S^N, S^1)$

This chapter is based on the paper *Topological singularities in* $W^{s,p}(S^N, S^1)$ accepted at the *Journal d'Analyse Mathématique*.

5.1 Introduction

In this article, we are interested in the location of the singularities of maps u defined on S^N with values into S^1. Assume first that $u \in C^\infty(S^N \setminus A, S^1) \cap W^{1,1}(S^N, S^1)$. When A is 'small' (i.e. of finite $(N-2)$ Hausdorff measure), the set A can be recovered from u by computing the Jacobian of u. This quantity has been introduced in [19] in the context of liquid cristals, and also studied in [52] and [2]. It is defined as follows: let ω_0 be the 1 form in $\mathbb{R}^2$ given by

$$\omega_0(y) := y_1 dy_2 - y_2 dy_1.$$

Its restriction to the unit circle is exactly the standard volume form on S^1. The pullback of ω_0 by u is defined by

$$u^\sharp \omega_0 := u_1 du_2 - u_2 du_1 =: j(u).$$

This definition makes sense not only when u is smooth (that is when $A = \emptyset$) but also when u belongs merely to $W^{1,1}(S^N, S^1)$. In this case, the Jacobian $J(u)$ of u will be defined, in the distribution sense, as $1/2d(u^\sharp\omega_0)$, that is:

$$\langle J(u), \omega\rangle = \frac{1}{2}\langle d(u^\sharp\omega_0), \omega\rangle := -\frac{1}{2}\langle u^\sharp\omega_0, \delta\omega\rangle, \quad \forall \omega \in C^\infty(\Lambda^2 S^N).$$

Here, $\langle ., .\rangle$ denotes the inner product between forms of the same degree and δ is the formal adjoint of the differential operator d. Using the Hodge operator $\star$ (see precise definitions in section 2), the Jacobian of u can also be written as:

$$\langle J(u), \omega\rangle = -\frac{1}{2}\int_{S^N} (u^\sharp\omega_0) \wedge (\star\delta\omega).$$

First, note that when u is smooth with values into S^1 (that is when $A = \emptyset$), the Jacobian $J(u)$ is zero, since we have in local coordinates:

$$\begin{aligned} J(u) = \frac{1}{2} d(u_1 du_2 - u_2 du_1) &= \frac{1}{2}(du_1 \wedge du_2 - du_2 \wedge du_1) \\ = du_1 \wedge du_2 &= \sum_{i<j} (u_{1x_i} u_{2x_j} - u_{1x_j} u_{2x_j}) dx_i \wedge dx_j . \end{aligned}$$

The rank of the tangent map $T_x u$ is at most 1, so that all the minors of order 2 vanish. This shows that $J(u)$ is zero when u is smooth.

Consider now the case when $N = 2$ and A is a nonempty finite set of points. Then (see [19] and also [9]), we have:

$$\star J(u) = \pi \sum_{a \in A} \deg(u, a) \delta_a , \tag{5.1}$$

where δ_a is the Dirac mass in a and $\deg(u, a)$ is the degree of the restriction of u to a small well-oriented circle around a.

When $N \geq 3$, there is an analogue of (5.1) provided A is a finite union of $N-2$ dimensional connected oriented boundaryless manifolds. Let C be any small circle which links with such a manifold, say Γ. On C there is a natural orientation which is consistent with the orientation of Γ. For any $u \in C^\infty(S^N \setminus \Gamma, S^1)$, we can define the degree of the restriction of u to C. This degree is independent of the choice of C (see a more precise statement in section 2) and we denote it by $\deg(u, \Gamma)$.

Then the value of $J(u)$ is given by the following proposition (stated in [2]):

Proposition 5.1 *When A is a smooth oriented $N-2$ dimensional boundaryless manifold ($N \geq 3$), with connected components $A_1, ..., A_r$, we have*

$$\star J(u) := \pi \sum_{i=1}^{r} deg(u, A_i) \int_{A_i} \cdot \tag{5.2}$$

Here, $\displaystyle\int_{A_i} \cdot$ is the $N-2$ current defined on the set of smooth forms of degree $N-2$ by: $\zeta \mapsto \displaystyle\int_{A_i} \zeta$ and $deg(u, A_i)$ is the degree of u around A_i.

Note that there exist topological obstructions on A and the degrees. For instance, when $N = 2$, $\langle J(u), 1 \rangle = 0$ (by definition of $J(u)$) so that

$$\sum_{a \in A} \deg(u, a) = 0.$$

The interest of $J(u)$ is the possibility to identify a singular set A which is still relevant for any map $u \in W^{1,1}(S^N, S^1)$. Indeed, let $\mathcal{R}_0$ be the following set:

$$\bullet N = 2 : \mathcal{R}_0 := \{u \in \bigcap_{1 \leq r < 2} W^{1,r}(S^2, S^1); u \text{ is smooth outside}$$
$$\text{a finite set of points}\}$$

$$\bullet N \geq 3 : \mathcal{R}_0 := \{u \in \bigcap_{1\leq r<2} W^{1,r}(S^N, S^1); u \text{ is smooth outside}$$
$$\text{a smooth oriented } N-2 \text{ dimensional boundaryless submanifold}\}.$$

The class $\mathcal{R}_0$ is dense in $W^{1,1}(S^N, S^1)$ (see [7]). Furthermore, J is a continuous map from $W^{1,1}(S^N, S^1)$ into $(W^{1,\infty}(\Lambda^2 S^N))^*$, the dual space of Lipschitz forms of degree 2 on S^N. Using these two results together, we get (see [22] for the case $N = 2$ and [2] for $N \geq 3$):

• $N = 2, \star J(u) = \pi \sum(\delta_{P_i} - \delta_{N_i})$ with $\sum_i d(P_i, N_i) \leq C||du||_{L^1(\Lambda^1 S^2)}$.

• $N \geq 3, \star J(u) = \pi\partial S$ where S is an $N-1$ dimensional rectifiable current (in the sense of [34]) whose mass $||S||$ satisfies $||S|| \leq C||du||_{L^1(\Lambda^1 S^N)}$.

There exists a converse to the previous properties (see [22] and [2]):

• $N = 2$, let $T := \sum(\delta_{P_i} - \delta_{N_i})$ with $\sum_i d(P_i, N_i) < \infty$. Then there exists $u \in W^{1,1}(S^N, S^1)$ such that $\star J(u) = \pi T$.

• $N \geq 3$, let T be the boundary of an $N-1$ dimensional rectifiable current with finite mass. Then there exists $u \in W^{1,1}(S^N, S^1)$ such that $\star J(u) = \pi T$.

To see that $J(u)$ does describe in some sense the singular set of u, the following result, due to Bethuel, is relevant:

$$u \in \overline{C^\infty(S^N, S^1)}^{W^{1,1}} \iff J(u) = 0. \tag{5.3}$$

The aim of this paper is twofold: we want to describe the range of $J(u)$ when u belongs to a fractional Sobolev space $W^{s,p}(S^N, S^1)$, and to generalise (5.3) to this context.

Let us first note that $C^\infty(S^N, S^1)$ is dense in $W^{s,p}(S^N, S^1)$ when $sp < 1$ (see [31]) or $sp \geq 2$ (see [15] when $N = 2$ and [8] when $N \geq 3$), and thus there is no 'good' notion of singular set in that case. Hence, in the following, we will assume that $1 \leq sp < 2$. If $s \geq 1$, then $W^{s,p}(S^N, S^1) \subset W^{1,1}(S^N, S^1)$, so that $J(u)$ is defined as above. In particular, it is still true that $\star J(u)$ is the boundary of a rectifiable current with codimension 1 and finite mass. However, such a current is *not* in general the Jacobian of some $u \in W^{s,p}(S^N, S^1)$. A counterexample is given at the beginning of section 3.

Let $\mathcal{E}$ denote the set of $N-2$ currents of the form:

$$\bullet N = 2 : \quad \pi \sum_{i=1}^{r} (\delta_{B_i} - \delta_{C_i}) \ , \ r \in \mathbb{N}, \text{ where } B_i, C_i \text{ are points in } S^2,$$

$$\bullet N \geq 3 : \quad \pi \sum_{i=1}^{r} \int_{A_i} \cdot \ , \ r \in \mathbb{N}, \text{ where } A_i \text{ is a smooth oriented connected}$$

$N-2$ dimensional boundaryless submanifold.

Our main result is the following:

Theorem 5.1 *Let $s \geq 1, 1 \leq p < \infty, 1 < sp < 2$.*

a) If u belongs to $W^{s,p}(S^N, S^1)$, then $\star J(u)$ belongs to the closure of $\mathcal{E}$ in $W^{s-2,p}(\Lambda^{N-2} S^N) \cap W^{-1,sp}(\Lambda^{N-2} S^N)$. Moreover, we have

$$||J(u)||_{W^{s-2,p}(\Lambda^2 S^N)} \leq C||u||_{W^{s,p}(S^N)} \ , \ ||J(u)||_{W^{-1,sp}(\Lambda^2 S^N)} \leq C||u||^{1/s}_{W^{s,p}(S^N)}.$$

b) Conversely, if M belongs to the closure of $\mathcal{E}$ in

$$W^{s-2,p}(\Lambda^{N-2} S^N) \cap W^{-1,sp}(\Lambda^{N-2} S^N),$$

then there exists $u \in W^{s,p}(S^N, S^1)$ such that $\star J(u) = M$. In addition, we may choose u such that

$$||u||_{W^{s,p}(S^N)} \leq C(||M||_{W^{s-2,p}(\Lambda^{N-2}S^N)} + ||M||^{s}_{W^{-1,sp}(\Lambda^{N-2}S^N)})$$

for some constant $C \geq 0$.

To prove this theorem, we will use a density result:

Theorem 5.2 *The set $\mathcal{R} := \mathcal{R}_0 \cap W^{s,p}(S^N, S^1)$ is dense in $W^{s,p}(S^N, S^1)$.*

This answers an open problem raised in [14]. Theorem 5.2 was already known for $s = 1$ (see [7]), and $s < 1$ (see [10], which generalizes previous results in [77], [41]). Our result covers the remaining case $1 < s$.

Finally, the analogue of (5.3) in the context of $W^{s,p}(S^N, S^1)$ spaces is

Theorem 5.3

$$u \in \overline{C^\infty(S^N, S^1)}^{W^{s,p}(S^N,S^1)} \Longleftrightarrow J(u) = 0.$$

In the case when $s < 1$, the Jacobian can still be defined, but with another formula (see [10]). The description of $J(u)$ in that case remains open. However, Theorem 5.3 still holds when $N = 2$ and $s < 1$ (see [75]).

The paper is organized as follows. In the next section, we describe the notations and give the precise definitions used throughout the article. In section 3, we prove Proposition 5.1 and the first part of Theorem 5.1. The proof relies on the regularity theory for the Laplace-Beltrami operator (briefly recalled in the last section) and the density of $\mathcal{R}$ (whose proof is postponed to section 5). Section 4 is dedicated to the proof of the second part of Theorem 5.1 and to the proof of Theorem 5.3.

5.2 Definitions

The unit sphere S^N is a smooth manifold of dimension N, embedded in $\mathbb{R}^{N+1}$, and it inheritates from $\mathbb{R}^{N+1}$ its Riemannian structure and its orientation (via its outer normal).

The Riemannian metric gives birth to an inner product on any tangent space T_xS^N to S^N at $x \in S^N$. We will denote it by $(.|.)$ (without mentioning the dependence on x). It can be extended to antisymetric multilinear forms on T_xS^N with the same notation. Then, we can define an inner product on l forms $(0 \leq l \leq N)$ as

$$\langle \alpha, \beta \rangle := \int_{S^N} (\alpha_x|\beta_x) d\mathcal{H}^N(x)$$

for any $\alpha, \beta \in C^\infty(\Lambda^l S^N)$, that is the set of smooth l forms on S^N. This inner product will be extended to measurable forms as soon as $x \to (\alpha_x|\beta_x)$ is an integrable function on S^N.

We follow [34] for the definitions of the exterior differential d, the codifferential δ and the Hodge operator. In particular, the Hodge operator $\star$ is a map from the l forms onto the $N-l$ forms $(0 \le l \le N)$ such that if $(e_1, ..., e_N)$ is an oriented orthonormal basis on $T_x S^N$, then

$$\star e_\alpha = \sigma(\alpha, \bar{\alpha}) e_{\bar{\alpha}}$$

where $\alpha = (\alpha_1 < ... < \alpha_l), e_\alpha = e_{\alpha_1} \wedge ... \wedge e_{\alpha_l}, \bar{\alpha}$ is the complement of α in $[|1, N|]$ in the natural increasing order and $\sigma(\alpha, \bar{\alpha})$ is the sign of the permutation which reorders $(\alpha, \bar{\alpha})$ in the natural increasing order. Then

$$\star\star = (-1)^{l(N-l)}$$

on l forms. We will use the fact that:

$$\langle \alpha, \beta \rangle = \int_{S^N} \alpha \wedge (\star \beta) \ , \ \forall \alpha, \beta \in C^\infty(\Lambda^l S^N).$$

The codifferential operator δ maps the smooth l forms $C^\infty(\Lambda^l S^N)$ into the smooth $l-1$ forms $C^\infty(\Lambda^{l-1} S^N)$. It is the formal adjoint of the differential operator d, that is:

$$\langle \delta\alpha, \beta \rangle = -\langle \alpha, d\beta \rangle, \ \ \forall \alpha \in C^\infty(\Lambda^l S^N), \ \beta \in C^\infty(\Lambda^{l-1} S^N).$$

The following property will be often used:

$$\delta = (-1)^{N(l+1)} \star d \star .$$

The Laplace-Beltrami operator on $C^\infty(\Lambda^l S^N)$ is

$$\Delta := d\delta + \delta d.$$

We need to define the degree of u around a smooth oriented connected $N-2$ dimensional boundaryless submanifold, say Γ. Fix $x_0 \in \Gamma$. There exists a connected neighborhood U of x_0 in Γ and two smooth vector fields v_1, v_2 on S^N such that $(v_1(x), v_2(x))$ is an orthonormal basis of $(T_x\Gamma)^\perp$ for any $x \in U$ (actually, this property could be assumed on the whole Γ since the normal bundle of an $N-2$ dimensional oriented boundaryless submanifold is trivial, see [66]). We may assume that $(v_1(x), v_2(x))$ is 'well-oriented', i.e. that, when $(e_1, .., e_{N-2})$ is a well-oriented basis of $T_x\Gamma$, then $(e_1, .., e_{N-2}, v_1(x), v_2(x))$ is a well-oriented basis of $T_x S^N$.

There exists $\eta > 0$ such that the endpoint $e(x, t_1, t_2)$ of the geodesic segment of length $r := (t_1^2 + t_2{}^2)^{1/2}$ which starts at x with the initial velocity vector $(t_1/r)v_1(x) + (t_2/r)v_2(x)$ is well defined for any $r < \eta$. Then, the map

$$e : (x, t_1, t_2) \in U \times B_{\mathbb{R}^2}(0, \eta) \mapsto e(x, t_1, t_2)$$

is a diffeomorphism from $U \times B_{\mathbb{R}^2}(0, \eta)$ onto a neighborhood U_η of U in S^N (see the Product Neighborhood Theorem, [68]). Now, for any $x \in U$, we can define the circle $C(x, r)$ centered in x and of radius $r < \eta$ as the set

$$C(x, r) := \{e(x, r\cos\theta, r\sin\theta) : \theta \in [0, 2\pi]\}.$$

We define the degree of u on $C(x, r)$ as the degree of the map $v : S^1 \to S^1, v(\cos\theta, \sin\theta) := u(e(x, r\cos\theta, r\sin\theta))$. Note that the parametrization $\theta \mapsto$

$e(e, r\cos\theta, r\sin\theta)$ defines an orientation on $C(x,r)$, and that the degree of u on $C(x,r)$ is precisely the degree of u with respect to this orientation.

We next check that this degree does not depend on x and on small $r > 0$. Let $(x,r),(x',r') \in U \times [0,\eta)$. We want to show that there exists an orientation preserving homotopy which maps continuously $C(x,r)$ onto $C(x',r')$. Since Γ is connected, there exists a continuous map $l : [0,1] \to \Gamma$ such that $l(0) = x$ and $l(1) = x'$. Then, we define:

$$H : (t,\theta) \in [0,1] \times [0,2\pi] \to e(l(t), [(1-t)r + tr']\cos\theta, [(1-t)r + tr']\sin\theta).$$

The map H is the desired homotopy. By connectedness, it does make sense to define the degree $\deg(u,\Gamma)$ of u as the degree of u restricted to $C(x,r)$ for any $x \in \Gamma$ and any r sufficently small.

Let $(U_i', V_i', \phi_i)_{i\in\{1,2\}}$ be an oriented atlas of S^N and $U_i \subset \overline{U}_i \subset U_i'$ be open sets such that $U_1 \cup U_2 = S^N$. We denote $V_i := \phi_i(U_i)$. Let $(\theta_i)_{i\in\{1,2\}}$ be a partition of unity subordinate to the covering $(U_i)_{i\in\{1,2\}}$. We will also introduce $\psi_i = \phi_i^{-1}$. We will denote by

$$g_{jk}(x) := (\frac{\partial}{\partial x_j} | \frac{\partial}{\partial x_k})$$

the coefficients of the metric tensor of g (in local coordinates $(x_1, ..., x_N) := \phi_i$) and $(g^{jk}(x)) = (g_{jk}(x))^{-1}$. By continuity and compacity, there exists $C > 0$ such that

$$||d_x\phi_i|| \leq C, ||d_y\psi_i|| \leq C, \frac{1}{C}|\eta|^2 \leq \sum_{j,k} g_{jk}(x)\eta_j\eta_k \leq C|\eta|^2$$

for any $i = 1,2, x \in U_i, y \in V_i, \eta = (\eta_1, ..., \eta_N) \in \mathbb{R}^N$.

The space of l currents is the topological dual of the space of l forms: $C^\infty(\Lambda^l S^N)$, the latter being equipped with the usual topology, see [81]. It will be denoted by $\mathcal{D}'(\Lambda^l S^N)$. Any integrable l form $\alpha \in L^1(\Lambda^l S^N)$ defines an l current by:

$$\langle T_\alpha, \beta\rangle := \int_{S^N} (\alpha_x|\beta_x) d\,\mathcal{H}^N(x) \quad , \ \forall \beta \in C^\infty(\Lambda^l S^N). \tag{5.4}$$

In the following, we will identify α and T_α. This identification is a guideline to define several operations on currents. For instance,

$$\langle \star T, \omega\rangle = (-1)^{l(N-l)}\langle T, \star\omega\rangle$$

for any $\omega \in C^\infty(\Lambda^l S^N)$. The exterior differential d as well as the codifferential δ are defined by duality on $\mathcal{D}'(\Lambda^l S^N)$.

The multiplication of a distribution on l forms $T \in \mathcal{D}'(\Lambda^l(M))$ and a smooth function θ is defined as:

$$\langle \theta T, \alpha\rangle := \langle T, \theta\alpha\rangle, \quad \forall \alpha \in C^\infty(\Lambda^l S^N).$$

The pushing forward of a distribution $T \in \mathcal{D}'(\Lambda^l(S^N))$ compactly supported in some U_i by the smooth diffeomormphism $\phi_i : U_i \to V_i$ is defined by

$$\langle \phi_{i\sharp} T, \alpha\rangle = \langle \star T, \phi_i^\sharp(\star_0\alpha)\rangle, \quad \forall \alpha \in C^\infty(\Lambda^l V_i),$$

where $\star_0$ is the Hodge operator in $\mathbb{R}^N$ (endowed with the Euclidean metric) and $\phi_i^\sharp(\star_0\alpha)$ denotes the pullback of $\star_0\alpha$ by ϕ_i.

To justify this definition, note that if $T = T_\omega$ were defined by an integrable l form ω, as in (5.4), then we would set $\phi_{i\sharp}T_\omega := T_{\phi_{i\sharp}\omega}$, that is for any $\alpha \in C^\infty(\Lambda^l V_i)$:

$$\begin{aligned} \langle \phi_{i\sharp}T_\omega, \alpha\rangle = \int_{V_i} (\phi_{i\sharp}\omega|\alpha)_0 &= \int_{V_i} (\phi_{i\sharp}\omega) \wedge (\star_0\alpha) \\ = \int_{U_i} \phi_i^\sharp\{(\phi_{i\sharp}\omega) \wedge (\star_0\alpha)\} &= \int_{U_i} \omega \wedge \phi_i^\sharp(\star_0\alpha) = \langle \star T_\omega, \phi_i^\sharp(\star_0\alpha)\rangle. \end{aligned}$$

(In the first line, we have denoted by $(\cdot|\cdot)_0$ the Euclidean inner product on $\mathbb{R}^N$).

Note also that since $\phi_{i\sharp}T$ is compactly supported in V_i (its support being included in $\phi_i(\mathrm{supp}T)$), we can consider it as an element of $\mathcal{D}'(\Lambda^l\mathbb{R}^N)$.

The multiplication of a distribution by an element of the partition of unity is called *localization*. The pushing forward of a distribution by ϕ_i is called *rectification*. Finally, when a distribution is compactly supported in an open set $V \subset \mathbb{R}^N$, we will automatically identify it with a distribution on $\mathbb{R}^N$, in the usual way. This procedure corresponds to the one described in the case of 0 forms in [86].

Several spaces of functions, of forms, of distributions on forms appear in the statement of the theorems or in the proofs below. Sobolev spaces on l forms $(0 \le l \le N)$ $W^{k,p}(\Lambda^l S^N), k \in \mathbb{N}, p \ge 1$ are defined as in [71], Chapter 7 (or [34]), that is *via* charts defining an atlas on S^N. In [82], one can find an *intrinsic* definition of Sobolev spaces on forms (that is without references to local charts), which turns out to be rather convenient. When $1 < p < \infty$ and $k \in \mathbb{N}^*$, we define $W^{-k,p}(\Lambda^l S^N) := (W^{k,p'}(\Lambda^l S^N))^*$, where $p' = p/(p-1)$. Besov spaces of functions and of distributions on the boundary of an open set (which is the case of S^N) are defined in [86], and some properties of these sets are studied there. We will denote them $B^s_{p,q}(S^N), s \in \mathbb{R}, p, q \ge 1$. The corresponding definitions for p forms and distributions on p forms (which could be called *Besov currents*) remain to be given, thanks to a localization-rectification procedure.

Let $A(\mathbb{R}^N)$ be a vector subspace of $\mathcal{D}'(\mathbb{R}^N)$, equipped with a norm $||.||_{A(\mathbb{R}^N)}$. We make two hypotheses on $A(\mathbb{R}^N)$: the *multiplication property* and the *diffeomorphism property*. The multiplication property requires that for any $u \in A(\mathbb{R}^N)$ and any $\theta \in C^\infty_c(\mathbb{R}^N), \theta u \in A(\mathbb{R}^N)$ with $||\theta u||_{A(\mathbb{R}^N)} \le C(\theta)||u||_{A(\mathbb{R}^N)}$. The diffeomorphism property requires that for any $u \in A(\mathbb{R}^N)$ compactly supported in some open set V and for any diffeomorphism ϕ between two open sets U and V in $\mathbb{R}^N$, the distribution $u \circ \phi$ belongs to $A(\mathbb{R}^N)$ and satisfies $||u \circ \phi||_{A(\mathbb{R}^N)} \le C(\phi)||u||_{A(\mathbb{R}^N)}$.

Now, it is possible to define $A(\Lambda^l\mathbb{R}^N)$ as the product of l copies of $A(\mathbb{R}^N)$, endowed with the product topology (and a norm defining it). This definition follows the definition of $\mathcal{D}'(\Lambda^l\mathbb{R}^N)$, the set of distributions on l forms, which can be identified with the product of l copies of $\mathcal{D}'(\mathbb{R}^N)$. Then $A(\Lambda^l\mathbb{R}^N)$ still satisfies the multiplication property and the diffeomorphism property (where the multiplication and the composition are now understood in the sense of l currents $\mathcal{D}'(\Lambda^l\mathbb{R}^N)$, exactly as we have done above in the case of S^N).

Finally, we define $A(\Lambda^l S^N)$ as the set of those elements T in $\mathcal{D}'(\Lambda^l S^N)$ such that for $i = 1, 2, \phi_{i\sharp}(\theta_i T) \in A(\Lambda^l\mathbb{R}^N)$. (Recall that $\phi_{i\sharp}(\theta_i T)$ is extended by 0 on

$\mathbb{R}^N \setminus V_i$). A norm on $A(\Lambda^l S^N)$ is then given by

$$\sum_i ||\phi_{i\sharp}(\theta_i T)||_{A(\Lambda^l\mathbb{R}^N))}.$$

Different atlases and partitions of unity yield equivalent norms.

The Besov spaces $B^s_{p,q}(\mathbb{R}^N)$ (see [86]) satisfy the multiplication property and the diffeomorphism property, so that we can define $B^s_{p,q}(\Lambda^l S^N)$, the Besov space of l forms on S^N.

Among the Besov spaces, only the fractional Sobolev spaces and their duals will be of interest to us. When s is not an integer, we set $W^{s,p}(\Lambda^l S^N) := B^s_{p,p}(\Lambda^l S^N)$.

For the following, it is also convenient to have intrinsic definitions of the space $W^{s,p}(S^N)$ when $s \in]1,2[$. We can see that $u \in W^{s,p}(S^N)$ if and only if

$$u \in W^{1,p}(S^N) \text{ and } D_{\sigma,p} du \in L^p(S^N)$$

where $\sigma := s-1$ and

$$D_{\sigma,p}\alpha(x) := \{\int_{S^N} \frac{|\alpha_x - \alpha_y|^p}{d(x,y)^{N+\sigma p}}\, dy\}^{1/p} \quad \forall \alpha \in L^p(\Lambda^1 S^N),$$

with $|\alpha_x - \alpha_y|$ defined by

$$|\alpha_x - \alpha_y| := \sum_{i:x,y\in U_i} |\alpha_x - \alpha_y|_i \tag{5.5}$$

and if $x, y \in U_i$,

$$|\alpha_x - \alpha_y|_i = \sum_{k=1}^N |\alpha^k(x) - \alpha^k(y)|$$

where $\alpha =: \sum_k \alpha^k dx_k$ in the local coordinates $(x_1, ..., x_N) := \phi_i$ on U_i. Then, for any $\alpha \in W^{\sigma,p}(\Lambda^1 S^N)$, we define

$$||\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} := ||\alpha||_{L^p(\Lambda^1 S^N)} + ||D_{\sigma,p} du||_{L^p(S^N)}.$$

Now, a norm on $W^{s,p}(S^N)$ is given by

$$||u||_{W^{s,p}(S^N)} := ||u||_{L^p(S^N)} + ||du||_{W^{\sigma,p}(\Lambda^1 S^N)}.$$

We will also use the notation $D_{\sigma,p}$ for functions $u \in L^p(S^N)$:

$$D_{\sigma,p}u(x) := \{\int_{S^N} \frac{|u(x) - u(y)|^p}{d(x,y)^{N+\sigma p}}\, dy\}^{1/p}$$

or for 1 forms with values into some $\mathbb{R}^d$ (if $\alpha := (\alpha_1, .., \alpha_d)$, the quantity $|\alpha_x - \alpha_y|$ becomes $\displaystyle\sum_{i:x,y\in U_i} \sum_{k=1}^N \sum_{j=1}^d |\alpha^k_j(x) - \alpha^k_j(y)|_i$).

The following remarks will be useful: The operator d is a bounded linear operator from $W^{s,p}(\Lambda^l S^N)$ into $W^{s-1,p}(\Lambda^{l+1} S^N)$, for $1 < p < \infty, s \in \mathbb{Z}$ or $1 \le p < \infty, s \notin \mathbb{Z}$. The multiplication property implies that if $T \in W^{s,p}(\Lambda^l S^N)$ and $\theta \in C^\infty(S^N)$, then $\theta T \in W^{s,p}(\Lambda^l S^N)$. Any embedding between two Besov spaces on $\mathbb{R}^N$ has its counterpart for Besov currents on S^N.

5.3 Proof of Theorem 5.1, first part

In this section, we want to prove Theorem 5.1 a). First, we are going to justify its interest by presenting an example of some $T \in \star J(W^{1,1}(S^N, S^1))$ which does not belong to $\star J(W^{s,p}(S^N, S^1))$. We consider the case $s = 1, p \in]1, 2[$ and $N = 2$. In that case, we know that

$$\star J(W^{1,1}(S^2, S^1)) := \{\pi \sum_i (\delta_{P_i} - \delta_{N_i}) : \sum_i d(P_i, N_i) < \infty\}.$$

Moreover, it is easy to see that $J(W^{1,p}(S^2, S^1)) \subset W^{-1,p}(\Lambda^2 S^2)$ (see details below).

Let $d_i := 1/i^{1/\alpha}$ where $\alpha \in]1 - 1/p', 1[$. Let $N_i := (\sqrt{1 - d_i^2}, 0, d_i)$ and $P_i := (\sqrt{1 - 4d_i^2}, 0, 2d_i)$. Set $T := \sum_i (\delta_{P_i} - \delta_{N_i})$. For any $n \geq 1$, we define $u_n(x, y, z) = z^\alpha$ if $z > 1/n$ and $1/n^\alpha$ elsewhere. Then, u_n is Lipschitz on S^2. The sequence $(||u_n||_{W^{1,p'}(S^2)})_n$ is bounded (here, we use $(1 - \alpha)p' < 1$). Hence, if T were in $W^{-1,p}(S^2)$, then the sequence $(|T(u_n)|)_n$ would be bounded too. We now show that this is not the case.

First, we note that if $0 < z_1 < z_2$, then

$$z_2^\alpha - z_1^\alpha \geq \alpha(z_2 - z_1)^\alpha (\frac{z_2 - z_1}{z_2})^{1-\alpha}.$$

This implies that, if $d_i \geq 1/n$, then

$$u_n(P_i) - u_n(N_i) \geq \alpha 2^{\alpha-1} d_i^\alpha,$$

so that

$$T(u_n) \geq \alpha 2^{\alpha-1} \sum_{i:d_i \geq 1/n} d_i^\alpha = \alpha 2^{\alpha-1} \sum_{i \leq n^\alpha} 1/i.$$

The right side goes to $+\infty$, as claimed. This completes the proof of the fact that $J(W^{1,p}(S^2, S^1))$ is strictly contained in $J(W^{1,1}(S^2, S^1))$.

To prove Theorem 5.1, we will first calculate J on the set $\mathcal{R}$ (Proposition 5.1): the result is well known but to our knowledge, no proof has been published yet. Then, we will show that J is continuous from $W^{s,p}(S^N, S^1)$ into $W^{s-2,p}(\Lambda^2 S^N) \cap W^{-1,sp}(\Lambda^2 S^N)$. Finally, we will use the density of $\mathcal{R}$ into $W^{s,p}(S^N, S^1)$ (the proof of which is postponed to section 6) to get the result.

Proof of Proposition 5.1. In the case when $N = 2$, a proof can be found in [9]. Hence, we restrict our attention to the case $N \geq 3$. Let Γ be a smooth oriented $N-2$ dimensional boundaryless submanifold of S^N. Let u be a smooth map on $S^N \setminus \Gamma$, and we assume that u belongs to $W^{1,1}(S^N, S^1)$. We want to prove that:

$$\langle J(u), \zeta \rangle = \pi \sum_{i=1}^r \deg(u, \Gamma_i) \int_{\Gamma_i} \star\zeta\ , \quad \forall \zeta \in C^\infty(\Lambda^2 S^N), \tag{5.6}$$

where $\Gamma_1, .., \Gamma_r$ are the connected components of Γ. As stated in section 2, there exist two smooth vector fields v_1, v_2 on S^N such that $(v_1(x), v_2(x))$ is an orthonormal basis of $(T_x\Gamma)^\perp$ for any $x \in \Gamma$. In addition, we may assume that (v_1, v_2) is well-oriented. There exists $\eta > 0$ such that the endpoint $e(x, t_1, t_2)$ of the geodesic segment of length $r := (t_1^2 + {t_2}^2)^{1/2}$ which starts at x with the initial velocity vector $(t_1/r)v_1(x) + (t_2/r)v_2(x)$ is well defined for any $r < \eta$ and the map

$$e : (x, t_1, t_2) \in \Gamma \times B_{\mathbb{R}^2}(0, \eta) \mapsto e(x, t_1, t_2)$$

is a diffeomorphism from $\Gamma \times B_{\mathbb{R}^2}(0, \eta)$ onto a neighborhood Δ_η of Γ. Each point $x \in \Gamma$ belongs to the domain U of a well-oriented chart $\phi_0 : U \subset S^N \to V \subset \mathbb{R}^N$ which satisfies:

$$\phi_0(U \cap \Gamma) = V \cap (\mathbb{R}^{N-2} \times \{(0, 0)\}).$$

We can assume that $U \subset \Delta_\eta$. We define:

$$\phi : x \in U \mapsto (\phi_0(x'), t_1, t_2) \in \mathbb{R}^{N-2} \times B_{\mathbb{R}^2}(0, \eta)$$

where $x' \in \Gamma, (t_1, t_2) \in B_{\mathbb{R}^2}(0, \eta)$ are defined by $e(x', t_1, t_2) = x$. Then ϕ is still a diffeomorphism from U onto $\phi(U)$ and we can assume (by shrinking U if necessary) that V has the form $]-\sigma, \sigma[^N$. The interest of this modification is that ϕ^{-1} maps the circle $C(\phi(x'), r) := \{(\phi(x'), r\cos\theta, r\sin\theta) : \theta \in [0, 2\pi]\}$ onto the circle in $S^N : \{e(x', r\cos\theta, r\sin\theta) : \theta \in [0, 2\pi]\}$. This remark will be useful below.

Let $\zeta \in C^\infty(\Lambda^2 S^N)$. Using a partition of unity, we may assume that ζ is compactly supported in the domain U of a chart ϕ of the type above.

In particular, $\operatorname{supp}\zeta$ intersects only one connected component of Γ, say Γ_1. Let us introduce some notations. We will decompose any $x \in \mathbb{R}^N$ as $x = (x', y, z) \in \mathbb{R}^{N-2} \times \mathbb{R} \times \mathbb{R}$. For small $\epsilon > 0$ and $\delta \in]0, \pi/2[$, we define:

$$\begin{aligned}
\Delta_\epsilon &:= \phi^{-1}(\{(x', y, z) \in V : |(y, z)| < \epsilon\}),\\
\Sigma_\epsilon &:= \phi^{-1}(\{(x', y, z) \in V : |(y, z)| = \epsilon\}),\\
\Sigma_{\epsilon,\delta} &:= \phi^{-1}(\{(x', \epsilon\cos\theta, \epsilon\sin\theta) \in V : \theta \in]\delta, 2\pi - \delta[\}),\\
A &:= \phi^{-1}(\{(x', y, z) \in V : z = 0, y \geq 0\}).
\end{aligned}$$

The set $U_0 := U \setminus A$ is simply connected (since it is homeomorphic to a star-shaped open set in $\mathbb{R}^N$). The map u is smooth on U_0 and takes its values into S^1. So, there exists some smooth function $\kappa : U_0 \to \mathbb{R}$ such that

$$u = (\cos\kappa, \sin\kappa) \quad \text{on} \quad U_0.$$

Moreover, $|\nabla\kappa| = |\nabla u|$, so that κ is Lipschitz continuous on $U_0 \cap \Sigma_\epsilon$, its Lipschitz constant depending only on ϵ. This implies that $\kappa \circ \phi^{-1}(x', \epsilon\cos\delta, \epsilon\sin\delta)$ has a limit $\kappa \circ \phi^{-1}(x', \epsilon, 0^+)$ when $\delta \to 0^+$, the convergence being uniform with respect to $x' \in]-\sigma, \sigma[^{N-2}$. Similarly, $\kappa \circ \phi^{-1}(x', \epsilon\cos\delta, \epsilon\sin\delta)$ converges to $\kappa \circ \phi^{-1}(x', \epsilon, 2\pi^-)$ when $\delta \to 2\pi^-$, uniformly with respect to x'. Furthermore, the quantity $\kappa \circ \phi^{-1}(x', \epsilon, 2\pi^-) - \kappa \circ \phi^{-1}(x', \epsilon, 0^+)$ is exactly $2\pi\deg(u, \Gamma_1)$ since

$$\phi^{-1}(\{(x', \epsilon\cos\theta, \epsilon\sin\theta) : \theta \in [0, 2\pi]\})$$

is the circle perpendicular to Γ_1 at x with radius ϵ. The definition of the Jacobian and the dominated convergence theorem imply that:

$$\langle J(u), \zeta\rangle = \lim_{\epsilon \to 0} \frac{1}{2} \int_{S^N \setminus \Delta_\epsilon} j(u) \wedge (d \star \zeta) = \lim_{\epsilon \to 0} \frac{1}{2} \int_{U \setminus \Delta_\epsilon} j(u) \wedge (d \star \zeta).$$

Using the formula $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$ for two forms α, β, we have:

$$\begin{aligned}
\int_{U \setminus \Delta_\epsilon} j(u) \wedge (d \star \zeta) &= -\int_{U \setminus \Delta_\epsilon} d(j(u) \wedge (\star\zeta)) + \int_{U \setminus \Delta_\epsilon} d(j(u)) \wedge (\star\zeta) \\
&= \int_{\partial(U \setminus \Delta_\epsilon)} j(u) \wedge (\star\zeta).
\end{aligned}$$

The second line follows from the Stokes' formula and the fact that $d(j(u)) = 0$ pointwise on $U \setminus \Delta_\epsilon$.

On U_0, we have $j(u) = d\kappa$. Whence (note that $\Sigma_{\epsilon,0} = \partial\Delta_\epsilon \setminus A$),

$$\int_{\partial(U \setminus \Delta_\epsilon)} j(u) \wedge (\star\zeta) = \lim_{\delta \to 0} \int_{\Sigma_{\epsilon,\delta}} d\kappa \wedge (\star\zeta).$$

Write once again:

$$\begin{aligned}
\int_{\Sigma_{\epsilon,\delta}} d\kappa \wedge (\star\zeta) &= \int_{\Sigma_{\epsilon,\delta}} d(\kappa(\star\zeta)) - \int_{\Sigma_{\epsilon,\delta}} \kappa d(\star\zeta) \\
&= \int_{\partial\Sigma_{\epsilon,\delta}} \kappa(\star\zeta) - \int_{\Sigma_{\epsilon,\delta}} \kappa d(\star\zeta).
\end{aligned}$$

We have:

$$\int_{\partial\Sigma_{\epsilon,\delta}} \kappa(\star\zeta) = \int_{S_{\epsilon,\delta}} \kappa(\star\zeta) + \int_{S_{\epsilon,2\pi-\delta}} \kappa(\star\zeta),$$

where

$$S_{\epsilon,\delta} := \phi^{-1}(\{(x', \epsilon\cos\delta, \epsilon\sin\delta) \in V\})$$

is oriented by $\Sigma_{\epsilon,\delta}$. Let us write explicitly the first quantity $\int_{S_{\epsilon,\delta}} \kappa(\star\zeta)$:

$$-\int_{]-\sigma,\sigma[^{N-2}} \kappa \circ \phi^{-1}(x', \epsilon\cos\delta, \epsilon\sin\delta)\phi_\sharp(\star\zeta)(x', \epsilon\cos\delta, \epsilon\sin\delta)\, dx'.$$

As explained above, the quantity under the sign $\int$ converges uniformly with respect to $x' \in]-\sigma,\sigma[^{N-2}$ when $\delta \to 0$ (and ϵ is fixed) to

$$\kappa \circ \phi^{-1}(x', \epsilon, 0^+)\phi_\sharp(\star\zeta)(x', \epsilon, 0).$$

So, we have:

$$\lim_{\delta \to 0} \int_{\partial\Sigma_{\epsilon,\delta}} \kappa(\star\zeta) = \int_{]-\sigma,\sigma[^{N-2}} \phi_\sharp(\star\zeta)(x', \epsilon, 0)(\kappa(x', \epsilon, 2\pi^-) - \kappa(x', \epsilon, 0^+))\, dx'$$

$$= 2\pi \deg(u, \Gamma_1) \int_{]-\sigma,\sigma[^{N-2}} \phi_\sharp(\star\zeta)(x', \epsilon, 0)\, dx'.$$

Before letting ϵ go to 0, it remains to estimate

$$\int_{\Sigma_{\epsilon,\delta}} \kappa d(\star\zeta).$$

This quantity is not greater than $||d\zeta||_{L^\infty(U)}||\kappa||_{L^1(\Sigma_\epsilon)}$, and

$$||\kappa||_{L^1(\Sigma_\epsilon)} \leq C\int_{]-\sigma,\sigma[^{N-2}} dx' \int_0^{2\pi} \epsilon\kappa\circ\phi^{-1}(x', \epsilon\cos\theta, \epsilon\sin\theta)d\theta.$$

We claim that this last quantity goes to 0. Let us admit this claim for a moment and complete the proof. We have

$$\int_{\partial(U\setminus\Delta_\epsilon)} j(u)\wedge(\star\zeta) = 2\pi\mathrm{deg}\,(u,\Gamma_1)\int_{]-\sigma,\sigma[^{N-2}} \phi_\sharp(\star\zeta)(x',\epsilon,0)\,dx' + o(1).$$

When ϵ goes to 0, we obtain:

$$\begin{aligned}\langle J(u),\zeta\rangle &= \pi\mathrm{deg}\,(u,\Gamma_1)\int_{]-\sigma,\sigma[^{N-2}} \phi_\sharp(\star\zeta)(x',0,0)\,dx' \\ &= \pi\mathrm{deg}\,(u,\Gamma_1)\int_{\Gamma_1}\star\zeta,\end{aligned}$$

which was required.

Let us now prove the claim. It amounts to proving the following result.

Lemma 5.1 *Let $v\in W^{1,1}(\mathbb{R}^N)$. Let $\Xi_\epsilon := \{(x',y,z) : |(y,z)| = \epsilon\}$. Then, $||v||_{L^1(\Xi_\epsilon)}$ goes to 0 when ϵ goes to 0.*

Proof: Let $Z_\epsilon := \{(x',y,z) : |(y,z)| < \epsilon\}$. The Stokes' formula implies (with ν the outing unit normal to Ξ_ϵ):

$$\begin{aligned}\int_{\Xi_\epsilon}|v| = \int_{\Xi_\epsilon}|v|\nu.\nu = \int_{Z_\epsilon}\mathrm{div}\,(|v|\nu) = \int_{Z_\epsilon}|v|\mathrm{div}\,\nu + \nabla|v|.\nu \\ = \int_{Z_\epsilon}\frac{|v|}{(y^2+z^2)^{1/2}} + \nabla|v|.\nu \leq \int_{Z_\epsilon}\frac{|v|}{(y^2+z^2)^{1/2}} + |\nabla v|.\end{aligned}$$

So, it is enough to show that $|v|/(y^2+z^2)^{1/2}$ is summable on Z_1. This follows from the above computation with $\epsilon = 1$. This completes the proof of Proposition 5.1.

□

We now show the following:

Proposition 5.2 *The operator J is continuous from $W^{s,p}(S^N, S^1)$ into*

$$W^{s-2,p}(S^N)\cap W^{-1,sp}(S^N).$$

This proposition relies on the multiplication properties of the fractional Sobolev spaces. To show some of them, we will have a frequent use of the following lemma (where $\sigma := s-1 \in]0,1[$).

Lemma 5.2 *([67]) Let $w\in W^{1,p}(S^N)$. Then there exists some constant $C\geq 0$ such that for almost every $x\in S^N$, we have*

$$D_{\sigma,p}w(x) \leq C(\mathcal{M}|w-w(x)|^p(x))^{(1-\sigma)/p}(\mathcal{M}|dw|^p(x))^{\sigma/p}.$$

Here, $\mathcal{M}$ denotes the maximal function

$$\mathcal{M}|dw|^p(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|dw|^p(y)\,dy.$$

Corollary 5.1 *There exists $C > 0$ such that:*
a) For any $w \in W^{1,sp}(S^N, B_{\mathbb{R}^2}(0,3))$ and $z \in L^{sp}(S^N)$, we have:

$$||zD_{\sigma,p}w||_{L^p(S^N)} \leq C||z||_{L^{sp}(S^N)}||dw||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}.$$

b) For any $w \in W^{1,sp}(S^N, B_{\mathbb{R}^2}(0,3))$ and $\alpha \in L^{sp}(\Lambda^1 S^N) \cap W^{\sigma,p}(\Lambda^1 S^N)$, we have:

$$\begin{aligned} ||w\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} &\leq ||w\alpha||_{L^p(\Lambda^1 S^N)} + ||\alpha||_{L^{sp}(\Lambda^1 S^N)}||D_{\sigma,p}w||_{L^{sp/\sigma}(S^N)} \\ &\quad +||wD_{\sigma,p}\alpha||_{L^p(S^N)} \\ &\leq C||\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} + C||\alpha||_{L^{sp}(\Lambda^1 S^N)}||dw||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}. \end{aligned}$$

c) For any $w \in W^{s,p}(S^N, B_{\mathbb{R}^2}(0,3))$ and $\alpha \in L^{sp}(\Lambda^1 S^N) \cap W^{\sigma,p}(\Lambda^1 S^N)$, we have:

$$||w\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq C||\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} + C||\alpha||_{L^{sp}(\Lambda^1 S^N)}||w||^{\sigma/s}_{W^{s,p}(S^N)}.$$

Proof: Part a) follows from Hölder's inequality and the boundedness of $\mathcal{M}$ on L^s:

$$||zD_{\sigma,p}w||_{L^p(S^N)} \leq ||z||_{L^{sp}(S^N)}||D_{\sigma,p}w||_{L^{s'p}(S^N)} \text{ , with } s' = s/(s-1)$$

$$\leq C||z||_{L^{sp}(S^N)}||w||^{1-\sigma}_{L^{\infty}(S^N)}||\mathcal{M}|dw|^p||^{\sigma/p}_{L^s(S^N)}$$

$$\leq C||z||_{L^{sp}(S^N)}||dw||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}.$$

We now prove part b).

$$||w\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq ||w\alpha||_{L^p(\Lambda^1 S^N)} + ||D_{\sigma,p}(w\alpha)||_{L^p(S^N)}$$

$$\leq ||w\alpha||_{L^p(\Lambda^1 S^N)} + |||\alpha|D_{\sigma,p}w||_{L^p(S^N)} + ||wD_{\sigma,p}\alpha||_{L^p(S^N)}$$

$$\leq ||w\alpha||_{L^p(\Lambda^1 S^N)} + ||\alpha||_{L^{sp}(\Lambda^1 S^N)}||D_{\sigma,p}w||_{L^{s'p}(S^N)} + ||wD_{\sigma,p}\alpha||_{L^p(S^N)}$$

$$\leq ||w||_{L^{\infty}(S^N)}||\alpha||_{W^{\sigma,p}(\Lambda^1 S^N)} + C||\alpha||_{L^{sp}(\Lambda^1 S^N)}||dw||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}$$

(this is the same calculation as in part a).

Part c) follows from part a) thanks to the inequality:

$$||u||_{W^{1,sp}(S^N)} \leq C||u||^{1/s}_{W^{s,p}(S^N)}||u||^{1-1/s}_{L^{\infty}(S^N)}. \tag{5.7}$$

(see [78], Theorem 2.2.5). This completes the proof of the corollary.

□

Let $u = (u^1, u^2) \in W^{s,p}(S^N, S^1)$. Then $du^2 \in W^{\sigma,p}(\Lambda^1 S^N) \cap L^{sp}(\Lambda^1 S^N)$. Corollary 5.1c shows that $u^1 du^2 \in L^{sp}(\Lambda^1 S^N) \cap W^{\sigma,p}(\Lambda^1 S^N)$. Hence, $j(u)$ lies in this space so that finally, $J(u) = dj(u) \in W^{-1,sp}(\Lambda^2 S^N) \cap W^{s-2,p}(\Lambda^2 S^N)$.

If a sequence (u_n) converges in $W^{s,p}(S^N, S^1)$ to some u, let us prove that $J(u_n)$ converges to $J(u)$ in $W^{-1,sp}(\Lambda^2 S^N)$ and in $W^{s-2,p}(\Lambda^2 S^N)$.

First, we show that $u_n^{\sharp}\omega_0$ converges to $u^{\sharp}\omega_0$ in $L^{sp}(\Lambda^1 S^N)$. This will imply the convergence of $J(u_n)$ to $J(u)$ in $W^{-1,sp}(\Lambda^2 S^N)$ since d is continuous from $L^{sp}(\Lambda^1 S^N)$ into $W^{-1,sp}(\Lambda^2 S^N)$. Now,

$$||u_n^1 du_n^2 - u^1 du^2||_{L^{sp}(\Lambda^1 S^N)} \leq ||(u_n^1 - u^1)du_n^2||_{L^{sp}(\Lambda^1 S^N)} + ||du_n^2 - du^2||_{L^{sp}(\Lambda^1 S^N)}$$

since $|u|=1$. The second term goes to 0 because of the continuous embedding $W^{s,p}(\Lambda^1 S^N,S^1)\subset W^{1,sp}(\Lambda^1 S^N,S^1)$. Up to a subsequence, we can assert the existence of a $k\in L^1(S^N)$ such that $|du_n|^{sp}\le k$ almost everywhere, and the convergence almost everywhere of u_n^1 to u^1. The dominated convergence theorem implies that for this subsequence, the first term in the right hand side goes to 0. Actually, this argument is valid for any subsequence of the original sequence u_n, that is, from any subsequence of the sequence $||(u_n^1-u^1)du_n^2||_{L^{sp}(\Lambda^1 S^N)}$, we can extract a subsequence which converges to 0. This shows that the whole original sequence goes to 0. Similarly, $||u_n^2 du_n^1-u^2du^1||_{L^{sp}(\Lambda^1 S^N)}$ converges to 0. So $J(u_n)$ converges to $J(u)$ in $W^{-1,sp}(\Lambda^2 S^N)$.

We have now to prove that $u_n^\sharp\omega_0$ converges to $u^\sharp\omega_0$ in $W^{\sigma,p}(\Lambda^1 S^N)$ (this will imply the convergence of $J(u_n)$ to $J(u)$ in $W^{s-2,p}(\Lambda^2 S^N)$). Thanks to Corollary 5.1a and c, we have:

$$\begin{aligned}
||u_n^1du_n^2-u^1du^2||_{W^{\sigma,p}(\Lambda^1 S^N)} &\le ||(u_n^1-u^1)du^2||_{W^{\sigma,p}(\Lambda^1 S^N)}\\
&\quad +||u_n^1(du_n^2-du^2)||_{W^{\sigma,p}(\Lambda^1 S^N)}\\
&\le ||(u_n^1-u^1)D_{\sigma,p}(du^2)||_{L^p(S^N)}+|||du^2|D_{\sigma,p}(u_n^1-u^1)||_{L^p(S^N)}\\
&\quad +||u_n^1(du_n^2-du^2)||_{W^{\sigma,p}(\Lambda^1 S^N)}+||(u_n^1-u^1)du^2||_{L^p(\Lambda^1 S^N)}\\
&\le ||(u_n^1-u^1)D_{\sigma,p}(du^2)||_{L^p(S^N)}+C||du^2||_{L^{sp}(\Lambda^1 S^N)}||du_n^1-du^1||^{\sigma}_{L^{sp}(\Lambda^1 S^N)}\\
&\quad +C||du_n^2-du^2||_{W^{\sigma,p}(\Lambda^1 S^N)}+C||du_n^2-du^2||_{L^{sp}(\Lambda^1 S^N)}||u_n^1||^{\sigma/s}_{W^{s,p}(S^N)}\\
&\quad +||(u_n^1-u^1)du^2||_{L^p(\Lambda^1 S^N)}.
\end{aligned}$$

The right hand side goes to 0 (use the dominated convergence theorem for the terms $||(u_n^1-u^1)D_{\sigma,p}(du_2)||_{L^p(S^N)}$ and $||(u_n^1-u^1)du^2||_{L^p(\Lambda^1 S^N)}$).

This completes the proof of the continuity of J, which implies Theorem 5.1 a, in view of the calculation of J on $\mathcal{R}$ (at the beginning of this section) and the density of $\mathcal{R}$ (see section 5).

□

5.4 Proof of Theorem 5.1, part 2

The second part of Theorem 5.1 is a consequence of the following lemma:

Lemma 5.3 *Let Γ be a smooth oriented $(N-2)$ dimensional boundaryless submanifold of $S^N, N\ge 3$. Let $\Gamma_1,..,\Gamma_r$ be its connected components and $a_1,..,a_r$ be integers. We define the 2 current T as:*

$$\langle T,\omega\rangle := \sum_{i=1}^r a_i\int_{\Gamma_i}\star\omega,\quad \forall\omega\in C^\infty(\Lambda^2 S^N). \tag{5.8}$$

Then there exists $u\in C^\infty(S^N\setminus\Gamma,S^1)\cap W^{s,p}(S^N,S^1)$ such that

$$J(u)=\pi T.$$

Moreover, we may choose u such that

$$||u||_{W^{s,p}(S^N)}\le C(||T||^s_{W^{-1,sp}(\Lambda^2 S^N)}+||T||_{W^{s-2,p}(\Lambda^2 S^N)}) \tag{5.9}$$

for some $C>0$ independent of Γ and of the a_i's.

Remark 5.1 *We have stated the lemma for the case $N \geq 3$. A similar statement holds for $N = 2$, with $\Gamma := \{A_1, .., A_r\} \subset S^N, a_1, .., a_r \in \mathbb{Z}$ such that $\sum_{i=1}^{r} a_i = 0$ and $\langle T, \omega \rangle := \sum_{i=1}^r a_i \star \omega(A_i)$. With minor modifications, our proof applies also to the case $N = 2$. We treat below only the case $N \geq 3$.*

Note that (5.9) is meaningful, since T belongs to both $W^{-1,sp}(\Lambda^2 S^N)$ and $W^{s-2,p}(\Lambda^2 S^N)$. Indeed, for any $\alpha \in W^{1,q}(\Lambda^2 S^N) \cap W^{2-s,p'}(\Lambda^2 S^N)$ (with $q = sp/(sp-1)$ and $p' = p/(p-1)$), we have (as a consequence of the trace theory and the fact that $q > 2$ and $2 - s - 2/p' > 0$):

$$|\int_\Gamma \star\alpha| \leq C||\star\alpha||_{L^1(\Lambda^{N-2}\Gamma)} \leq C||\star\alpha||_{W^{1-2/q,q}(\Lambda^{N-2}\Gamma)} \leq C||\alpha||_{W^{1,q}(\Lambda^2 S^N)}$$

$$\text{and} \quad |\int_\Gamma \star\alpha| \leq C||\star\alpha||_{L^1(\Lambda^{N-2}\Gamma)} \leq C||\star\alpha||_{W^{2-s-2/p',p'}(\Lambda^{N-2}\Gamma)}$$
$$\leq C||\alpha||_{W^{2-s,p'}(\Lambda^2 S^N)}.$$

We admit Lemma 5.3 for an instant and we prove Theorem 5.1 b). Let T be in the closure of the set of 2 currents $\star\mathcal{E}$ associated to a smooth connected $N-2$ dimensional boundaryless submanifold as in (5.8). Then, there exists a sequence $(T_n)_{n\in\mathbb{N}}$ satisfying the hypotheses of the lemma, converging in $W^{-1,sp}(\Lambda^2 S^N) \cap W^{s-2,p}(\Lambda^2 S^N)$ to T. The above lemma implies the existence of a sequence $(u_n)_{n\in\mathbb{N}}$, such that $J(u_n) = T_n$ and satisfying (5.9) with T replaced by T_n. The sequence (u_n) is bounded in $W^{s,p}(S^N, S^1) \subset W^{1,sp}(S^N, S^1)$. Then, up to a subsequence, we can assume that (u_n) converges a.e. to some $u \in W^{1,sp}(S^N, S^1)$, and since $|u_n| \leq 1$ a.e., the dominated convergence theorem shows that $(u_n)_{n\in\mathbb{N}}$ converges to u in L^q. We can also assume that $(du_n)_{n\in\mathbb{N}}$ weakly converges to du in $L^{sp}(\Lambda^1 S^N)$. Thus $(J(u_n))_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\Lambda^2 S^N)$ to $J(u)$. Hence $J(u) = \pi T$ and u satisfies (5.9).

Proof of Lemma 5.3: Let $M := S^N \setminus \Gamma$. Then M is a smooth open subset of S^N.

step 1: We first introduce $v \in W^{1,sp}(\Lambda^{N-2} S^N) \cap W^{s,p}(\Lambda^{N-2} S^N)$ such that $\delta dv = \star T = \gamma$ where γ denotes the $N-2$ current

$$\langle \gamma, \alpha \rangle = \sum_i a_i \int_{\Gamma_i} \alpha \quad , \ \forall \alpha \in C^\infty(\Lambda^{N-2} S^N).$$

Such a v exists. Indeed, Γ has no boundary, so that in the sense of distributions $\delta\gamma = 0$. This implies that γ vanishes on closed forms and thus on harmonic fields. Hence, denoting by $v := G(\gamma)$, (where G is the Green operator, see section 6), we have $\gamma = \delta dv + d\delta v = \delta dv$ since $0 = G(\delta\gamma) = \delta G(\gamma) = \delta v$. Moreover, as a consequence of the properties of the Green operator, the following estimates hold: there exists $C \geq 0$ such that:

$$||v||_{W^{s,p}(\Lambda^{N-2}S^N)} \leq C||\gamma||_{W^{s-2,p}(\Lambda^{N-2}S^N)} \leq C||T||_{W^{s-2,p}(\Lambda^2 S^N)}$$
$$||v||_{W^{1,sp}(\Lambda^{N-2}S^N)} \leq C||\gamma||_{W^{-1,sp}(\Lambda^{N-2}S^N)} \leq C||T||_{W^{-1,sp}(\Lambda^2 S^N)}.$$

Note that v is a measurable function, which is harmonic on M, and in particular smooth.

step 2: There exists an $N-1$ current A such that $\delta A = \gamma$; moreover, we may assume that for each i, there exists an $N-1$ dimensional rectifiable set A_i and a measurable $N-1$ form τ_i satisfying $|\tau_i| = 1$ a.e. such that

$$\langle A, \omega\rangle := \sum_i a_i \int_{A_i} (\omega|\tau_i)\, d\mathcal{H}^{N-1} \quad , \ \forall \omega \in C^\infty(\Lambda^{N-1}S^N).$$

Here, we use the fact that every rectifiable current in $\mathbb{R}^N$ with finite mass, bounded support and no boundary is the boundary of an integrable current with finite mass (see [2], Remark 2.6.).

We consider the 1 current $\star A$ defined by

$$\langle \star A, \alpha\rangle := (-1)^{N-1}\langle A, \star\alpha\rangle, \quad \forall \alpha \in C^\infty(\Lambda^1 S^N)$$

and set

$$C := \star dv - \star A.$$

We note that $dC := d\star(dv - A) = (-1)^{N-2}\star\delta(dv - A) = \star(\gamma-\gamma) = 0$. Then, thanks to a BV version of the Poincaré Lemma on manifolds (see Lemma 5.4 below), there exists some $\phi \in BV(S^N)$ such that (in the sense of distributions)

$$d\phi = C.$$

Lemma 5.4 *Let C be a 1 current on S^N such that $dC = 0$. We suppose that C is associated to a Radon measure on S^N, which means that*

$$sup\langle C, \alpha\rangle < +\infty$$

where the supremum is taken over all $\alpha \in C^\infty(\Lambda^1 S^N)$ satisfying

$$||\alpha||_{L^\infty(\Lambda^1 S^N)} \leq 1.$$

Then there exists $\phi \in BV(S^N)$ such that $d\phi = C$ (in the sense of distributions).

Proof: As usual, we regularize C, we apply the classical Poincaré Lemma to this smooth C and we then pass to the limit. We recall the following

Lemma 5.5 *([83]) For any p current D associated to a Radon measure on S^N and any $\epsilon > 0$, there exists $\omega_\epsilon \in C^\infty(\Lambda^{N-p}S^N)$ such that $\mathcal{R}_\epsilon(D)$ defined by*

$$\langle \mathcal{R}_\epsilon(D), \alpha\rangle = \int_{S^N} \omega_\epsilon \wedge \alpha \quad , \quad \forall \alpha \in C^\infty(\Lambda^p S^N)$$

satisfies:
i) $M(\mathcal{R}_\epsilon(D)) \leq (1+\epsilon)M(D)$ where $M(D) := \sup\langle D, \alpha\rangle$ over the maps $\alpha \in C^\infty(\Lambda^p S^N)$ satisfying $||\alpha||_{L^\infty(\Lambda^p S^N)} \leq 1$,
ii) if $\delta D = 0$ then $\delta\mathcal{R}_\epsilon(D) = 0$,
iii) $\mathcal{R}_\epsilon(D) \to D$ in $\mathcal{D}'(\Lambda^p S^N)$ when $\epsilon \to 0$.

Let $\beta_\epsilon \in C^\infty(\Lambda^{N-1}S^N)$ be such that

$$\langle \mathcal{R}_\epsilon(\star C), \alpha\rangle = \int_{S^N} (\beta_\epsilon|\alpha) d\mathcal{H}^N \ , \quad \forall \alpha \in C^\infty(\Lambda^{N-1}S^N).$$

Put it otherwise, β_ϵ is defined by $(-1)^{N-1} \star \beta_\epsilon := \omega_\epsilon$ where ω_ϵ is the 1 form appearing in the statement of Lemma 5.5 for $D := \star C$. Since $dC = 0$, we have $\delta\beta_\epsilon = 0$. Hence, by the classical version of the Poincaré Lemma, there exists a smooth function $\phi_\epsilon : S^N \to \mathbb{R}$ such that $\displaystyle\int_{S^N} \phi_\epsilon = 0$ and $d\phi_\epsilon = (-1)^{N-1} \star \beta_\epsilon$.

Then, using the Poincaré Sobolev inequality for $W^{1,1}$ functions,

$$\begin{aligned}
||\phi_\epsilon||_{L^1(S^N)} &\leq c\,||d\phi_\epsilon||_{L^1(\Lambda^1 S^N)} \\
&\leq c \sup_{||h||_{L^\infty(\Lambda^1 S^N)}\leq 1} \langle d\phi_\epsilon, h\rangle \leq c \sup_{||\alpha||_{L^\infty(\Lambda^{N-1} S^N)}\leq 1} \langle \beta_\epsilon, \alpha\rangle \\
&\leq c(1+\epsilon) \sup_{||h||_{L^\infty(\Lambda^1 S^N)}\leq 1} \langle C, h\rangle.
\end{aligned}$$

Hence, the sequence (ϕ_ϵ) is bounded in $W^{1,1}(S^N)$. Then, up to a subsequence, ϕ_ϵ converges in $BV(S^N)$ to a function of bounded variations ϕ. In particular, we have in the sense of distributions,

$$d\phi = \lim_{\epsilon\to 0} d\phi_\epsilon = \lim_{\epsilon\to 0} (-1)^{N-1} \star \beta_\epsilon = C.$$

step 3: Recall that, for any $f \in BV(S^N), df$ is the sum of three 1 currents of measure type: the absolutely continuous part $d_a f \llcorner \mathcal{H}^{\mathcal{N}}$, the Cantor part $d_C f$ which is singular with respect to the Lebesgue measure and does not charge any $\mathcal{H}^{N-1}$-finite set and the jump part $d_j f$ which is concentrated on a rectifiable set of codimension 1. Furthermore, $d_j f$ can be written as $[f]\nu_f \mathcal{H}^{N-1} \llcorner Sf$, where the $N-1$ rectifiable set Sf is the set of point of approximate discontinuity of f, ν_f is an $N-1$ form defining the orientation of Sf a.e. and the jump $[f]$ is the difference between the trace f^+ and f^- of f on the two sides of Sf (see [34] for details).

Here, we have

$$d\phi = d_a\phi + d_C\phi + d_j\phi = \star dv - \star A,$$

so that $d_C\phi = 0, d_a\phi = \star dv$ and $d_j\phi = -\star A$.

Since $d_j\phi = (\phi^+ - \phi^-)\nu_\phi \mathcal{H}^{N-1} \llcorner S\phi$, we see that $S\phi = \cup_i A_i$ $\mathcal{H}^{N-1}$ a.e. and that $\phi^+ - \phi^-$ is an integer $\mathcal{H}^{N-1}$ a.e. $x \in S\phi$.

step 4: Let us consider: $u := (-1)^N exp(2i\pi\phi)$.

Hence, thanks to the chain rule for BV functions (see [34]), u is a BV function with

$$d_a u = (-1)^N 2\pi i u d_a\phi = (-1)^N 2\pi i u \star dv \quad , \quad d_C u = 0$$

and $Su \subset S\phi$, with $(-1)^N(u^+ - u^-) = exp(2i\pi\phi^+) - exp(2i\pi\phi^-) = 0$ $\mathcal{H}^{N-1}$ a.e. $x \in Su$. Hence, $d_j u = 0$.

Thus $du = d_a u$ is absolutely continuous with respect to the Lebesgue measure.

step 5: Up to now, u is a smooth function on M. Moreover, since u is S^1 valued, $|du| \leq C|dv|$ so that

$$||du||_{L^{sp}(\Lambda^1 S^N)} \leq C||dv||_{L^{sp}(\Lambda^{N-2} S^N)} \leq C||T||_{W^{-1,sp}(\Lambda^2 S^N)}.$$

Let us now prove that

$$||du||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq C(||T||_{W^{\sigma-1,p}(\Lambda^2 S^N)} + ||T||^s_{W^{-1,sp}(\Lambda^2 S^N)}).$$

Thanks to Corollary 5.1b, we have (taking into account the fact that $|u| \leq 1$),

$$||du||_{W^{\sigma,p}(\Lambda^1 S^N)} \leq C||u \star dv||_{W^{\sigma,p}(\Lambda^1 S^N)}$$

$$\leq C(||dv||_{W^{\sigma,p}(\Lambda^{N-1} S^N)} + ||du||^{s-1}_{L^{sp}(\Lambda^1 S^N)}||dv||_{L^{sp}(\Lambda^{N-1} S^N)})$$

$$\leq C(||dv||_{W^{\sigma,p}(\Lambda^{N-1} S^N)} + ||dv||^{s-1}_{L^{sp}(\Lambda^{N-1} S^N)}||dv||_{L^{sp}(\Lambda^{N-1} S^N)})$$

$$\leq C(||T||_{W^{\sigma-1,p}(\Lambda^2 S^N)} + ||T||^s_{W^{-1,sp}(\Lambda^2 S^N)}).$$

Hence, $u \in W^{s,p}(\Lambda^1 S^N)$.

This ends the proof of Lemma 5.3, in view of the fact that:

$$J(u) = 1/2 du^\sharp \omega_0 = (-1)^N \pi d \star dv = \pi \star \delta dv = \pi \star \gamma = \pi T.$$

□

Proof of Theorem 5.3. If $u \in \overline{C^\infty(S^N, S^1)}^{W^{s,p}(S^N,S^1)}$, then there exists a sequence of smooth maps u_n converging to u in $W^{s,p}(S^N, S^1)$. Using the continuity of J from $W^{s,p}(S^N, S^1)$ into $\mathcal{D}'(\Lambda^2 S^N)$ and the fact that J vanishes on $C^\infty(S^N, S^1)$, we get $J(u) = 0$.

Conversely, if $J(u) = 0$ for some $u \in W^{s,p}(S^N, S^1)$, then there exists $\phi \in W^{s,p}(S^N) \cap W^{1,sp}(S^N)$ such that $j(u) = d\phi$. Indeed, there exists $k \in \mathbb{N}$ such that $G^k(j(u))$ (the k^{th} iterate of the Green operator) is C^1 on S^N (thanks to the Sobolev embeddings and in view of the regularization properties of the Green operator, see section 6). Moreover, $dG^k(j(u)) = G^k(dj(u)) = 0$. Then, by the smooth version of the Poincaré Lemma, there exists some $\psi \in C^1(S^N)$ such that $G^k(j(u)) = d\psi$. Then

$$j(u) = \Delta^k G^k(j(u)) = \Delta^k d\psi = d\Delta^k \psi.$$

Then, we set $\phi := \Delta^k \psi$. By construction and thanks to the regularization properties of the Green operator, ϕ is in $W^{s,p}(S^N) \cap W^{1,sp}(S^N)$.

So,

$$\begin{aligned} d(ue^{-i\phi}) &= e^{-i\phi}(du - iud\phi) = ue^{-i\phi}(\bar{u}du - iu^\sharp\omega_0) \\ &= ue^{-i\phi}(u_1 du_1 + u_2 du_2) = 1/2 ue^{-i\phi} d(u_1^2 + u_2^2) \\ &= 1/2 ue^{-i\phi} d1 = 0. \end{aligned}$$

Hence, there exists $C \in \mathbb{R}$ (since $|ue^{-i\phi}| = 1$) such that $u = e^{i(\phi + C)}$. Moreover, there exists a sequence of smooth functions $(\phi_n) \subset C^\infty(S^N)$ converging to ϕ in $W^{1,sp}(S^N) \cap W^{s,p}(S^N)$. Then, $u_n := e^{i\phi_n}$ converges to u in $W^{s,p}(S^N, S^1)$, see [21] and [67]. Finally, $u \in \overline{C^\infty(S^N, S^1)}^{W^{s,p}(S^N,S^1)}$.

□

5.5 The set $\mathcal{R}$ is dense in $W^{s,p}(S^N, S^1)$

The aim of this section is to prove Theorem 5.2. Let $s \geq 1, p \geq 1$ such that $1 \leq sp < 2$. The case $s = 1, p < 2$ of Theorem 5.2 has been proved in [7]. Then, we limit ourselves to the case $s \in]1, 2[, p \geq 1$, following the strategy of the proof of Lemma 23 in [9]. Recall that

$$\mathcal{R} := \{u \in \bigcap_{1 \leq r < 2} W^{1,r}(S^N, S^1) \cap W^{s,p}(S^N, S^1) : u \text{ is smooth outside}$$

$$\text{a smooth oriented } N-2 \text{ dimensional boundaryless submanifold}\}.$$

When $N = 2$, u is assumed to be smooth outside a finite set of points A in S^2.

We first introduce some notations. Let $f_a : \mathbb{R}^2 - \{a\} \to S^1$, be the function defined by:

$$f_a(X) := \frac{X - a}{|X - a|}$$

and $j_a : S^1 \to S^1$ the inverse of f_a when restricted to S^1.

For any $a \in B_{\mathbb{R}^2}(0, 1/10)$ and any $w : S^N \to \mathbb{R}^2$ we denote by w^a the map

$$w^a(x) := \frac{w(x) - a}{|w(x) - a|}$$

which is defined on $\{x \in S^N : w(x) \neq a\}$. We have

$$df_a(X) = \frac{Id}{|X - a|} - \frac{(X - a) \otimes (X - a)}{|X - a|^3}$$

where $(X - a) \otimes (X - a)$ denotes the 2×2 tensor $[(X - a) \otimes (X - a)]_{ij} = (X - a)_i (X - a)_j$, and for any smooth $w : S^N \to \mathbb{R}^2$ (or any $w \in W^{1,p}(S^N, S^1)$),

$$Dw^a(X) := \frac{Dw(X)}{|w(X) - a|} + \frac{(w(X) - a) \otimes (w(X) - a)}{|w(X) - a|^3} \cdot Dw(X)$$

for almost every $X \in \{X' \in S^N : w(X') \neq a\}$. Besides the fact that

$$|df_a(X)| \leq \frac{C}{|X - a|}, \tag{5.10}$$

we will also use the following Lipschitz property of df_a :

Lemma 5.6 *There exists $C \geq 0$ such that for any $X, Y \in \mathbb{R}^2 - \{a\}$,*

$$|df_a(X) - df_a(Y)| \leq C \frac{|X - Y|}{|X - a||Y - a|}. \tag{5.11}$$

Proof: First, remark that $df_a(X) = df_0(X - a)$ so that we can assume $a = 0$. Second, $df_0(\lambda X) = (1/\lambda) df_0(X)$ so that we can suppose $|X| = 1$. Finally, $df_0(R_\theta X) = R_\theta df_0(X) R_\theta^{-1}$ where R_θ is the rotation of angle θ. Hence, we may assume that $X = (1, 0), Y = (r \cos\theta, r \sin\theta)$. Then,

$$|df_a(X) - df_a(Y)| \leq C \frac{\max(|\sin\theta|, |r - \cos^2\theta|)}{r}.$$

We estimate the ratio $|\sin\theta|/|1-re^{i\theta}|$; the ratio $|r-\cos^2\theta|/|1-re^{i\theta}|$ is easier to handle. We have:

$$|1-re^{i\theta}| = \sqrt{(1-r)^2+2r(1-\cos\theta)} = |1-r|\sqrt{1+2r\frac{2\sin^2(\theta/2)}{(1-r)^2}}.$$

Then

$$\frac{|\sin\theta|}{|1-re^{i\theta}|} \leq \frac{\mu}{\sqrt{1+r\mu^2}} \quad \text{with} \quad \mu = \frac{2|\sin(\theta/2)|}{|1-r|}.$$

We have $\mu \leq 4$ if $r \leq 1/2$ and

$$\frac{\mu}{\sqrt{1+r\mu^2}} \leq \frac{\mu}{\sqrt{1+\mu^2/2}}$$

if $r > 1/2$. In any case $|\sin\theta|/|1-re^{i\theta}|$ is bounded independently of θ, r. The proof of Lemma 5.6 is complete.

□

The proof of Lemma 22 in [9] shows that

Claim 5.1 *For any smooth function $v : S^N \to B_{\mathbb{R}^2}(0,1)$ and for a.e. $a \in B_{\mathbb{R}^2}(0,1/10)$, the function v^a is smooth on $S^N \setminus v^{-1}(a)$ and belongs to $W^{1,r}$ for any $r < 2$.*

On $W^{s,p}(S^N, S^1)$, we choose the norm:

$$||u||_{W^{s,p}(S^N)} = ||u||_{L^p(S^N)} + ||du||_{L^p(\Lambda^1 S^N)} + ||D_{\sigma,p}du||_{L^p(S^N)},$$

with $\sigma = s-1$.

We will use the fact that

$$|d(u_1+u_2)_x - d(u_1+u_2)_y| \leq |du_{1x} - du_{1y}| + |du_{2x} - du_{2y}|,$$

(this is an easy consequence of the definition of $|\cdot|$, see section 2).

Let $u \in W^{s,p}(S^N, S^1)$. There exists a sequence of smooth functions $v_\epsilon : S^N \to B_{\mathbb{R}^2}(0,1)$ which converges to u in $W^{s,p}(S^N, \mathbb{R}^2)$. We can suppose further that v_ϵ converges to u $\mathcal{H}^N$ a.e. and that dv_ϵ converges to du $\mathcal{H}^N$ a.e. Using the continuous embedding $W^{s,p}(S^N) \cap L^\infty(S^N) \subset W^{1,sp}(S^N)$ (see (5.7)), we may also assume that the sequence (v_ϵ) converges to u in $W^{1,sp}(S^N)$. Note also that $j_a(u^a) = u$. We then set

$$u^a_\epsilon := j_a(v^a_\epsilon).$$

The proof of Lemma 22 in [9] shows that

Claim 5.2 *The quantity $\displaystyle\int_{B_{\mathbb{R}^2}(0,1/10)} ||u^a_\epsilon - u||^p_{W^{1,p}(S^N)}\, da$ converges to 0 when ϵ goes to 0.*

One of the main tool of the proof (that we omit here) is that when $p < 2$, there exists some $C \geq 0$ such that

$$\int_{B_{\mathbb{R}^2}(0,1/10)} \frac{da}{|X-a|^p} \leq C, \quad \forall\, |X| \leq 1.$$

The new result, which enables us to generalise the density theorem to the case $s > 1$ is the following claim.

Claim 5.3 *The quantity* $\int_{B_{\mathbb{R}^2}(0,1/10)} ||D_{\sigma,p}(du^a_\epsilon - du)||^p_{L^p(S^N)}\, da$ *converges to* 0 *when* ϵ *goes to* 0.

We admit Claim 5.3 for an instant and we complete the proof of Theorem 5.2. Let $l_\epsilon(a) := ||u^a_\epsilon - u||^p_{W^{s,p}(S^N)}$. We know that $l_\epsilon := \int_{B_{\mathbb{R}^2}(0,1/10)} l_\epsilon(a)\, da$ tends to 0 when ϵ goes to 0 thanks to Claim 5.2 and Claim 5.3. Since (Chebychev's inequality)

$$|\{a \in B_{\mathbb{R}^2}(0,1/10) : l_\epsilon(a) \geq \sqrt{l_\epsilon}\}| \leq \sqrt{l_\epsilon} \quad (\text{if} \quad l_\epsilon \neq 0),$$

we see that for each $\epsilon > 0$, there exists a regular value of v_ϵ, say a_ϵ, such that

$$l_\epsilon(a_\epsilon) \leq \sqrt{l_\epsilon}. \tag{5.12}$$

(By Sard's Theorem, almost every a is a regular value of v_ϵ.) For such an $a_\epsilon, u^{a_\epsilon}_\epsilon$ belongs to $W^{s,p}(S^N,S^1)$ and is smooth except on the smooth oriented $N-2$ dimensional boundaryless submanifold $v_\epsilon^{-1}(a_\epsilon)$ (respectively, a finite set of points when $N=2$). Hence, $u^{a_\epsilon}_\epsilon$ belongs to $\mathcal{R}$ and converges to u in $W^{s,p}(S^N)$.

We now prove Claim 5.3. We will denote $g_a := j_a \circ f_a : \mathbb{R}^2 - \{a\} \to S^1 \subset \mathbb{R}^2$. Note that $|dg_a(u(x)) - dg_a(u(y))|$ is well defined for almost every $x, y \in S^N$ via any norm on the set of linear maps from $\mathbb{R}^2$ into $\mathbb{R}^2$. Moreover,

$$D_{\sigma,p}(\alpha+\beta) \leq D_{\sigma,p}(\alpha) + D_{\sigma,p}(\beta)\ , \qquad \forall \alpha,\beta \in L^p(\Lambda^1 S^N, \mathbb{R}^2).$$

We find that for any regular value a of v_ϵ:

$$\begin{aligned}
||D_{\sigma,p}(d(g_a\circ u) - d(g_a \circ v_\epsilon))||_{L^p(S^N)} &= ||D_{\sigma,p}(dg_a(u)\circ du - dg_a(v_\epsilon)\circ dv_\epsilon)||_{L^p(S^N)} \\
&= ||D_{\sigma,p}\{(dg_a(u) - dg_a(v_\epsilon))\circ dv_\epsilon + dg_a(u)\circ(du - dv_\epsilon)\}||_{L^p(S^N)} \\
&\leq ||D_{\sigma,p}\{(dg_a(u) - dg_a(v_\epsilon))\circ dv_\epsilon\}||_{L^p(S^N)} + ||D_{\sigma,p}\{dg_a(u)\circ(du - dv_\epsilon)\}||_{L^p(S^N)} \\
&\leq |||dv_\epsilon|D_{\sigma,p}(dg_a(u) - dg_a(v_\epsilon))||_{L^p(S^N)} + |||dg_a(u) - dg_a(v_\epsilon)|D_{\sigma,p}(dv_\epsilon)||_{L^p(S^N)} \\
&\quad + |||du - dv_\epsilon|D_{\sigma,p}(dg_a(u))||_{L^p(S^N)} + |||dg_a(u)|D_{\sigma,p}(du - dv_\epsilon)||_{L^p(S^N)}.
\end{aligned}$$

The fourth term is lower than $||dg_a(u)||_\infty ||D_{\sigma,p}(du - dv_\epsilon)||_{L^p}$ which goes to 0 (recall that u is S^1 valued so that $||dg_a(u)||_\infty$ is lower than a constant independent from a). Let us denote by A_1, A_2, A_3 the three terms still to be estimated. We have

$$A_2^p \leq C\int_{|v_\epsilon|<1/2} |D_{\sigma,p}(dv_\epsilon)|^p \left(\frac{1}{|u-a|^p} + \frac{1}{|v_\epsilon - a|^p}\right)$$

$$+C\int_{|v_\epsilon|\geq 1/2} |D_{\sigma,p}(dv_\epsilon)|^p |dg_a(u) - dg_a(v_\epsilon)|^p =: C(B_1^p + B_2^p).$$

Since dv_ϵ converges to du in $W^{\sigma,p}(\Lambda^1 S^N)$, we find that $||D_{\sigma,p}(dv_\epsilon - du)||_{L^p(S^N)}$ goes to 0. Thus, there exists some $k_0 \in L^p(S^N)$ such that (up to a subsequence) $|D_{\sigma,p}(dv_\epsilon - du)| \leq k_0$. Hence, $D_{\sigma,p}(dv_\epsilon) \leq D_{\sigma,p}(dv_\epsilon - du) + D_{\sigma,p}(du)$ is lower than the L^p function $k := k_0 + D_{\sigma,p}(du)$. On the set where $|v_\epsilon| \geq 1/2, u$ and v_ϵ remain far from $B_{\mathbb{R}^2}(0,1/10)$, so that $|dg_a(u) - dg_a(v_\epsilon)|$ remains bounded.

Since $dg_a(v_\epsilon) \to dg_a(u)$ a.e., the dominated convergence theorem implies that $\int_{B_{\mathbb{R}^2}(0,1/10)} B_2^p \, da \to 0$ when $\epsilon \to 0$.

Furthermore,

$$\begin{aligned}
\int_{B_{\mathbb{R}^2}(0,1/10)} B_1^p &\leq C \int_{|v_\epsilon|<1/2} k^p \int_{B_{\mathbb{R}^2}(0,1/10)} \left(\frac{1}{|u-a|^p} + \frac{1}{|v_\epsilon - a|^p}\right) da \\
&\leq C \int_{|v_\epsilon|<1/2} k^p
\end{aligned}$$

which goes to 0 since $|\{|v_\epsilon| < 1/2\}|$ goes to 0 as $\epsilon \to 0$. Using Corollary 5.1a (with $z := |du - dv_\epsilon|$ and $w := dg_a(u)$), we see that

$$A_3 \leq C||d^2 g_a(u)||^\sigma_{L^\infty(S^N, \mathcal{L}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2))} ||du||^\sigma_{L^{sp}(\Lambda^1 S^N)} ||du - dv_\epsilon||_{L^{sp}(\Lambda^1 S^N)}.$$

Thus, $A_3 \to 0$ as $\epsilon \to 0$.

The term A_1 involves the most tricky computations. Let us introduce a smooth function $\psi : [0, \infty[\to [0,1]$ such that

$$\psi(t) = \begin{cases} 0 \text{ if } & t \leq 1/4, \\ 1 \text{ if } & t \geq 1/2. \end{cases}$$

We decompose $dg_a(v_\epsilon)$ as

$$dg_a(v_\epsilon) := dg_a(v_\epsilon)\psi(|v_\epsilon|) + dg_a(v_\epsilon)(1 - \psi(|v_\epsilon|)).$$

This decomposition yields

$$\begin{aligned}
A_1 &= |||dv_\epsilon| D_{\sigma,p}(dg_a(u) - dg_a(v_\epsilon))||_{L^p(S^N)} \\
&= |||dv_\epsilon| D_{\sigma,p}\{dg_a(u) - dg_a(v_\epsilon)\psi(|v_\epsilon|) - dg_a(v_\epsilon)(1 - \psi(|v_\epsilon|))\}||_{L^p(S^N)} \\
&\leq |||dv_\epsilon| D_{\sigma,p}\{dg_a(u) - dg_a(v_\epsilon)\psi(|v_\epsilon|)\}||_{L^p(S^N)} \\
&\quad + |||dv_\epsilon| D_{\sigma,p}\{dg_a(v_\epsilon)(1 - \psi(|v_\epsilon|))\}||_{L^p(S^N)} \\
&=: K_1 + K_2.
\end{aligned}$$

Using Corollary 5.1a with $z = |dv_\epsilon|$ and $w = dg_a(u) - dg_a(v_\epsilon)\psi(|v_\epsilon|)$, and the fact that dg_a is bounded near S^1, we obtain

$$\begin{aligned}
K_1 &\leq C||dv_\epsilon||_{L^{sp}(\Lambda^1 S^N)} ||d\{dg_a(u) - dg_a(v_\epsilon)\psi(|v_\epsilon|)\}||^\sigma_{L^{sp}(S^N)} \\
&\leq C||dv_\epsilon||_{L^{sp}(\Lambda^1 S^N)} \{||d^2 g_a(u) \circ du - d^2 g_a(v_\epsilon) \circ dv_\epsilon \psi(|v_\epsilon|)||^\sigma_{L^{sp}(S^N)} \\
&\quad + |||dg_a(v_\epsilon)||d(\psi \circ |v_\epsilon|)|||^\sigma_{L^{sp}(S^N)}\}.
\end{aligned}$$

The dominated convergence theorem shows that this quantity goes to 0 when ϵ goes to 0.

Next, we turn our attention to K_2.

$$\begin{aligned}
K_2^p &:= |||dv_\epsilon| D_{\sigma,p}\{dg_a(v_\epsilon)(1 - \psi(|v_\epsilon|))\}||^p_{L^p(S^N)} \\
&\leq \int_{|v_\epsilon(x)|<1/2} |dv_\epsilon(x)|^p (D_{\sigma,p}\{dg_a(v_\epsilon)(1 - \psi(|v_\epsilon|))\})^p \, dx \\
&\quad + \int \int_{|v_\epsilon(y)|<1/2} |dv_\epsilon(x)|^p \frac{|D|^p}{|d(x,y)|^{N+\sigma p}} \, dy \, dx
\end{aligned}$$

with
$$D := dg_a(v_\epsilon(x))(1-\psi(|v_\epsilon(x)|)) - dg_a(v_\epsilon(y))(1-\psi(|v_\epsilon(y)|)).$$
Writing $|dv_\epsilon(x)|^p \le 2^p(|dv_\epsilon(x)-dv_\epsilon(y)|^p + |dv_\epsilon(y)|^p)$, we get that
$$\int_{|v_\epsilon(y)|<1/2}\int |dv_\epsilon(x)|^p \frac{|D|^p}{|d(x,y)|^{N+\sigma p}}\,dx\,dy$$
is lower than $C(\xi+\zeta)$, where
$$\xi := \int_{|v_\epsilon(y)|<1/2}\int |dv_\epsilon(x)-dv_\epsilon(y)|^p \frac{|D|^p}{|d(x,y)|^{N+\sigma p}},$$
$$\zeta := \int_{|v_\epsilon(y)|<1/2}\int |dv_\epsilon(y)|^p \frac{|D|^p}{|d(x,y)|^{N+\sigma p}}.$$
Recalling that
$$|D| \le C(\frac{1}{|v_\epsilon(x)-a|^p} + \frac{1}{|v_\epsilon(y)-a|^p}),$$
we obtain
$$\int_{B_{\mathbb{R}^2}(0,1/10)} \xi(a)\,da \le C\int_{|v_\epsilon(y)|<1/2} |D_{\sigma,p} dv_\epsilon(y)|^p\,dy$$
which is lower than $\int_{|v_\epsilon(y)|<1/2} k^p(y)\,dy$. This last quantity converges to 0. Concerning ζ, we have:
$$\zeta = \zeta(a) = \int_{|v_\epsilon(x)|<1/2} |dv_\epsilon(x)|^p (D_{\sigma,p}\{dg_a(v_\epsilon)(1-\psi(|v_\epsilon|))\})^p\,dx.$$
It remains to show that $\int_{B_{\mathbb{R}^2}(0,1/10)} \zeta(a)\,da \to 0$.

For any $X, Y \in B_{\mathbb{R}^2}(0,1)\setminus\{a\}$, we have:
$$dg_a(X) - dg_a(Y) = (dj_a(f_a(X)) - dj_a(f_a(Y)))\circ df_a(X)$$
$$+(dj_a(f_a(Y)))\circ(df_a(X)-df_a(Y)).$$
Using Lemma 5.6 combined with the inequality
$$|f_a(X)-f_a(Y)| = |\frac{X-a}{|X-a|} - \frac{Y-a}{|Y-a|}| \le 2\frac{|X-a||Y-X|}{|X-a||Y-a|} = 2\frac{|X-Y|}{|Y-a|},$$
we find that
$$\begin{aligned} |dg_a(X)-dg_a(Y)| &\le C\frac{|f_a(X)-f_a(Y)|}{|X-a|} + C\frac{|X-Y|}{|X-a||Y-a|} \\ &\le C\frac{|X-Y|}{|X-a||Y-a|}. \end{aligned} \tag{5.13}$$
Moreover,
$$|(1-\psi(|v_\epsilon(x)|))dg_a(v_\epsilon(x)) - (1-\psi(|v_\epsilon(y)|))dg_a(v_\epsilon(y))| \le$$

$$2|dg_a(v_\epsilon(x)) - dg_a(v_\epsilon(y))| + |dg_a(v_\epsilon(y))||\psi(|v_\epsilon(x)|) - \psi(|v_\epsilon(y)|)|.$$

Thanks to the mean value inequality applied to ψ, we have:

$$|\psi(|v_\epsilon(x)|) - \psi(|v_\epsilon(y)|)| \leq C||v_\epsilon(x)| - |v_\epsilon(y)|| \leq C|v_\epsilon(x) - v_\epsilon(y)|,$$

so that:

$$\begin{aligned}|dg_a(v_\epsilon(y))||\psi(|v_\epsilon(x)|) - \psi(|v_\epsilon(y)|)| &\leq C\frac{|v_\epsilon(x) - v_\epsilon(y)|}{|v_\epsilon(y) - a|} \\ &\leq C\frac{|v_\epsilon(x) - v_\epsilon(y)|}{|v_\epsilon(x) - a||v_\epsilon(y) - a|}.\end{aligned}$$

Thanks to (5.13) with $X := v_\epsilon(x), Y := v_\epsilon(y)$, we have:

$$|dg_a(v_\epsilon(x)) - dg_a(v_\epsilon(y))| \leq C\frac{|v_\epsilon(x) - v_\epsilon(y)|}{|v_\epsilon(x) - a||v_\epsilon(y) - a|}.$$

Finally,

$$|(1 - \psi(|v_\epsilon(x)|))dg_a(v_\epsilon(x)) - (1 - \psi(|v_\epsilon(y)|))dg_a(v_\epsilon(y))| \leq$$

$$C\frac{|v_\epsilon(x) - v_\epsilon(y)|}{|v_\epsilon(x) - a||v_\epsilon(y) - a|}.$$

Hence,

$$\begin{aligned}&D_{\sigma,p}\{dg_a(v_\epsilon)(1 - \psi(|v_\epsilon|))\}(x)^p \\ &\leq C\int_{S^N} \frac{|v_\epsilon(y) - v_\epsilon(x)|^p}{d(x,y)^{N+\sigma p}|v_\epsilon(x) - a|^p|v_\epsilon(y) - a|^p}\, dy.\end{aligned}$$

So,

$$\zeta(a) \leq C\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\, dy|dv_\epsilon(x)|^p\frac{|v_\epsilon(y) - v_\epsilon(x)|^p}{d(x,y)^{N+\sigma p}|v_\epsilon(x) - a|^p|v_\epsilon(y) - a|^p}. \tag{5.14}$$

In the sequel, we will use the following lemma:

Lemma 5.7 *For any $X, Y \in B_{\mathbb{R}^2}(0,1)$, we have*

$$\int_{B_{\mathbb{R}^2}(0,1/10)} \frac{da}{|X - a||Y - a|} \leq C(1 + |\ln|X - Y||)$$

and $\displaystyle\int_{B_{\mathbb{R}^2}(0,1/10)} \frac{da}{|X - a|^p|Y - a|^p} \leq \frac{C}{|X - Y|^{2p-2}}$ *when $p > 1$.*

Proof: Suppose first that $p > 1$. Using the change of variables $a' = -X + a$ and then $a'' = a'/|Z|$ with $Z := Y - X$, we have

$$\begin{aligned}\int_{B_{\mathbb{R}^2}(0,1/10)} \frac{da}{|X - a|^p|Y - a|^p} &= \int_{B_{\mathbb{R}^2}(-X,1/10)} \frac{da'}{|a'|^p|Z - a'|^p} \\ &= \frac{1}{|Z|^{2(p-1)}}\int_{B_{\mathbb{R}^2}(-X/|Z|,1/(10|Z|))} \frac{da''}{|a''|^p|Z/|Z| - a''|^p} \\ &\leq \frac{1}{|Z|^{2(p-1)}}\int_{\mathbb{R}^2} \frac{da''}{|a''|^p|(1,0) - a''|^p}\end{aligned}$$

which completes the proof of the case $p > 1$ in view of the fact that

$$\int_{\mathbb{R}^2} \frac{da''}{|a''|^p|(1,0)-a''|^p} \le c\Big(\int_{B_{\mathbb{R}^2}(0,1/2)} \frac{da}{|a|^p} + \int_{B_{\mathbb{R}^2}(0,2)-B_{\mathbb{R}^2}(0,1/2)} \frac{da}{|a-(1,0)|^p}$$

$$+ \int_{B_{\mathbb{R}^2}(0,2)^c} \frac{da}{|a|^{2p}}\Big) < \infty.$$

When $p = 1$, the proof is the same apart from the last estimate:

$$\int_{B_{\mathbb{R}^2}(-\frac{X}{|Z|},\frac{1}{10|Z|})} \frac{da''}{|a''||Z/|Z|-a''|} \le C + C\int_{B_{\mathbb{R}^2}(-\frac{X}{|Z|},\frac{1}{10|Z|})\setminus B_{\mathbb{R}^2}(0,2)} \frac{da''}{|a''|^2}$$

$$\text{and} \quad \int_{B_{\mathbb{R}^2}(-\frac{X}{|Z|},\frac{1}{10|Z|})\setminus B_{\mathbb{R}^2}(0,2)} \frac{da''}{|a''|^2} \le \int_{B_{\mathbb{R}^2}(0,\frac{2}{|Z|})\setminus B_{\mathbb{R}^2}(0,2)} \frac{da''}{|a''|^2}$$

$$\le C(|\ln|Z|| + 1).$$

□

Using Lemma 5.7 in (5.14) for $X = v_\epsilon(x)$ and $Y = v_\epsilon(y)$, we get that $\int_{B_{\mathbb{R}^2}(0,1/10)} \zeta(a)\, da$ is not greater than

$$C\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\, dy |dv_\epsilon(x)|^p \frac{|v_\epsilon(x)-v_\epsilon(y)|^{2-p}}{d(x,y)^{N+\sigma p}}$$

when $p > 1$ and

$$C\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\, dy |dv_\epsilon(x)| \frac{|v_\epsilon(x)-v_\epsilon(y)|}{d(x,y)^{N+\sigma}}(1 + |\ln|v_\epsilon(x)-v_\epsilon(y)||)$$

when $p = 1$. In the latter case, the term

$$\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\, dy |dv_\epsilon(x)| \frac{|v_\epsilon(x)-v_\epsilon(y)|}{d(x,y)^{N+\sigma}}$$

can be easily handled using Corollary 5.1a while the term

$$\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\, dy |dv_\epsilon(x)| \frac{|v_\epsilon(x)-v_\epsilon(y)|}{d(x,y)^{N+\sigma}} |\ln|v_\epsilon(x)-v_\epsilon(y)||$$

is not greater than

$$C\int_{|v_\epsilon(x)|<1/2}\int_{S^N} dx\, dy |dv_\epsilon(x)| \frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma}}$$

for any $\alpha \in]0, 1-\sigma[$ and some $C = C(\alpha)$.

In any case, a variation on Lemma 5.2 implies that for any $\alpha \in]0, 1-\sigma p[$,

$$\int_{S^N} \frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}}\, dy \quad \le \quad c([\mathcal{M}(|dv_\epsilon|^{1-\alpha})(x)]^{\sigma p/(1-\alpha)} + 1). \quad (5.15)$$

To prove (5.15), we adapt an idea of Hedberg (see [48], see also [67]). There exists $\delta_0>0$ (independent of x) such that the exponential map $\exp_x$ is a smooth diffeomorphism from $B_{T_xS^N}(0,\delta_0)$ onto $B_{S^N}(x,\delta_0)$. Fix $\delta\in(0,\delta_0)$. First,

$$\int_{S^N\setminus B_{S^N}(x,\delta)} \frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}}\,dy \le \sum_{k=0}^{\infty}\int_{\delta\le\frac{d(x,y)}{2^k}<2\delta} \frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{(2^k\delta)^{N+\sigma p}}\,dy$$

$$\le C\sum_{k=0}^{\infty}\frac{(2^{k+1}\delta)^N}{(2^k\delta)^{N+\sigma p}}\frac{1}{|B_{S^N}(x,2^{k+1}\delta)|}\int_{B_{S^N}(x,2^{k+1}\delta)}|v_\epsilon-v_\epsilon(x)|^{1-\alpha}$$

$$\le C\delta^{-\sigma p}(\sum_{k=0}^{\infty}2^{-k\sigma p})\mathcal{M}|v_\epsilon-v_\epsilon(x)|^{1-\alpha}(x).$$

Furthermore, using the change of variable $y\mapsto k=(\exp_x)^{-1}(y)$, we get:

$$\int_{B_{S^N}(x,\delta)}\frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}}\,dy \le C\int_{B_{T_xS^N}(0,\delta)}\frac{|v_\epsilon(x)-v_\epsilon(\exp_x(k))|^{1-\alpha}}{||k||^{N+\sigma p}}\,dk$$

$$\le C\int_{B_{T_xS^N}(0,\delta)}\frac{dk}{||k||^{N+\sigma p}}\int_0^1|dv_\epsilon(\exp((1-t)k))|^{1-\alpha}||k||^{1-\alpha}\,dt$$

$$\le C\int_0^1\frac{dt}{(1-t)^{1-\alpha-\sigma p}}\int_{B_{S^N}(x,(1-t)\delta)}\frac{|dv_\epsilon(z)|^{1-\alpha}}{d(z,x)^{N+\alpha+\sigma p-1}}\,dz$$

$$\le C\sum_{k=0}^{\infty}\int_{2^{-k-1}\delta\le d(x,z)<2^{-k}\delta}\frac{|dv_\epsilon(z)|^{1-\alpha}}{d(z,x)^{N+\sigma p+\alpha-1}}\,dz$$

$$\le C\sum_{k=0}^{\infty}(\delta 2^{-k})^{1-\alpha-N-\sigma p}(\delta 2^{-k})^N\frac{1}{|B_{S^N}(x,\delta 2^{-k})|}\int_{B_{S^N}(x,\delta 2^{-k})}|dv_\epsilon(z)|^{1-\alpha}\,dz$$

$$\le C\delta^{1-\alpha-\sigma p}\mathcal{M}|dv_\epsilon|^{1-\alpha}(x).$$

Thus,

$$\int_{S^N}\frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}}\,dy$$

$$\le C(\delta^{1-\alpha-\sigma p}\mathcal{M}|dv_\epsilon|^{1-\alpha}(x)+\delta^{-\sigma p}\mathcal{M}|v_\epsilon-v_\epsilon(x)|^{1-\alpha}(x)).$$

Minimizing on $\delta\le\delta_0$, we get:

$$\int_{S^N}\frac{|v_\epsilon(x)-v_\epsilon(y)|^{1-\alpha}}{d(x,y)^{N+\sigma p}}\,dy$$

$$\le C(\mathcal{M}|dv_\epsilon|^{1-\alpha}(x))^{\sigma p/(1-\alpha)}(\mathcal{M}|v_\epsilon-v_\epsilon(x)|^{1-\alpha}(x))^{(1-\alpha-\sigma p)/(1-\alpha)}$$

$$+C\delta_0^{-\sigma p}(\mathcal{M}|v_\epsilon-v_\epsilon(x)|^{1-\alpha}(x)).$$

Using the fact that v_ϵ is uniformly bounded by 1, we get the expected result (5.15).

We now use (5.15) in the estimate of $\int_{B_{\mathbb{R}^2}(0,1/10)} \zeta(a)\,da$. When $p > 1$, we take $\alpha := p-1$. The map $\mathcal{M}$ being bounded on $L^{sp/(2-p)}$,

$$\begin{aligned} \int_{B_{\mathbb{R}^2}(0,1/10)} \zeta(a)\,da &\leq C||dv_\epsilon||^p_{L^{sp}(|v_\epsilon|<1/2)}(||\mathcal{M}|dv_\epsilon|^{2-p}||^{p\sigma/(2-p)}_{L^{sp/(2-p)}(S^N)}+1) \\ &\leq C||dv_\epsilon||^p_{L^{sp}(|v_\epsilon|<1/2)}(||dv_\epsilon||^{p\sigma}_{L^{sp}(\Lambda^1 S^N)}+1) \end{aligned}$$

which converges to 0 when ϵ goes to 0, thanks to the dominated convergence theorem. When $p = 1$, a similar estimate holds for any $\alpha \in]0, 1-\sigma[$. This completes the proof of Claim 5.3 and Theorem 5.2.

□

5.6 The Laplacian on S^N

In this final section, we describe and prove some results concerning the regularity of the solutions of:

$$\Delta v = T \tag{5.16}$$

to be solved in fractional Sobolev spaces $W^{s,p}(\Lambda^l S^N, S^1)$, with $s, p \geq 1, sp > 1$.

We recall here the main results, following Scott [82]. We will also prove few results, presumably well-known to experts, but that we could not find in the literature.

First, we define the harmonic l *fields* by

$$\mathcal{H}(\Lambda^l S^N) := \{h \in C^\infty(\Lambda^l S^N) : dh = \delta h = 0\}.$$

This is a finite dimensional vector space, whose orthogonal space (with respect to the inner product on l forms) will be denoted by $\mathcal{H}(\Lambda^l S^N)^\perp$. Then, we denote by $H(\omega)$ the harmonic projection into $\mathcal{H}(\Lambda^l S^N)$ of an l form ω, that is:

$$\langle \omega - H(\omega), h\rangle = 0$$

for any $h \in \mathcal{H}(\Lambda^l S^N)$. (In fact, $\mathcal{H}(\Lambda^l S^N) = \{0\}$ if $0 < l < N$. We have introduced these notations for the sake of generality, since all the results of this article can be generalized to the case when S^N is replaced by more general manifolds).

Now, (Definition 5.23 and Proposition 6.1 in [82]) for any $\omega \in L^p(\Lambda^l S^N)$, where $1 < p < \infty$, there exists some $G(\omega) \in W^{2,p}(\Lambda^l S^N) \cap \mathcal{H}(\Lambda^l S^N)^\perp$ such that

$$\Delta G(\omega) = \omega - H(\omega)$$

and G is a bounded linear operator from $L^p(\Lambda^l S^N)$ into $W^{2,p}(\Lambda^l S^N)$. Moreover, G is selfadjoint and commutes with the Laplacian, the differential and the codifferential.

The Green operator G and the harmonic projection H can be extended to $\mathcal{D}'(\Lambda^l S^N)$, by duality, setting $\langle G(\omega), \alpha\rangle = \langle \omega, G(\alpha)\rangle$ and the same for H. We still have $\Delta G(\omega) = \omega - H(\omega)$ for any $\omega \in \mathcal{D}'(\Lambda^l S^N)$.

By duality, G is also continuous from $W^{-2,p}(\Lambda^l S^N)$ into $L^p(\Lambda^l S^N), 1 < p < \infty$. Furthermore, if $T \in W^{-1,p}(\Lambda^l S^N)$ and $v := G(T)$, we already know that v

is in $L^p(\Lambda^l S^N)$, since $T \in W^{-2,p}(\Lambda^l S^N)$, and for any $\alpha \in L^{p'}(\Lambda^l S^N)$, we have $\delta\alpha = \delta\Delta G(\alpha) = \Delta\delta G(\alpha)$, so that

$$\begin{aligned}
\langle dv, \alpha\rangle &= -\langle v, \delta\alpha\rangle \\
&= -\langle v, \Delta(\delta G(\alpha))\rangle \\
&= -\langle T, \delta G(\alpha)\rangle \\
&\leq ||T||_{W^{-1,p}} ||\delta G(\alpha)||_{W^{1,p'}} \\
&\leq C||T||_{W^{-1,p}} (||d\delta G(\alpha)||_{L^{p'}} + ||\delta G(\alpha)||_{L^{p'}}) \quad \text{(see [82], Cor 4.12)} \\
&\leq C||T||_{W^{-1,p}} ||\alpha||_{L^{p'}} \quad \text{(see [82], Prop 5.15, Prop 5.17).}
\end{aligned}$$

This shows that $dv \in L^p(\Lambda^{l+1} S^N)$ and $||dv||_{L^p(\Lambda^{l+1}S^N)} \leq C||T||_{W^{-1,p}(\Lambda^l S^N)}$. We have a similar estimate for $||\delta v||_{L^p(\Lambda^{l-1}S^N)}$. Hence (see [82], Cor 4.12), G is a bounded linear operator from $W^{-1,p}(\Lambda^l S^N)$ into $W^{1,p}(\Lambda^l S^N)$.

When $s \notin \mathbb{Z}, 1 < p < \infty$, the fractional Sobolev spaces $W^{s,p}$ can be defined by interpolation (see [78]). If we combine this with the previous remarks, we have:

Proposition 5.3 *The Green operator G is a bounded linear operator from*

$$W^{s-2,p}(\Lambda^l S^N) \quad \textit{into} \quad W^{s,p}(\Lambda^l S^N),$$

when $0 \leq s \leq 2, 1 < p < \infty$.

The case $p = 1, 1 < s < 2$ is also needed and not covered by the previous proposition. This is the object of the remaining part of this section:

Theorem 5.4 *Fix $l \in [|0, N|]$ and $1 < s < 2$. There exists $C > 0$ such that for any $T \in W^{s-2,1}(\Lambda^l S^N)$ satisfying $H(T) = 0$, there is an $\omega \in W^{s,1}(\Lambda^l S^N)$ such that $\Delta\omega = T$ and*

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C||T||_{W^{s-2,1}(\Lambda^l S^N)}.$$

It is well-known that this statement is false for $s = 1$. To prove the theorem, we use the Besov's spaces and the fact that they coincide with Sobolev's spaces for noninteger values of s. Actually, the proof of Theorem 5.4 is true when $W^{s,1}$ is replaced by $W^{s,p}$ for any $1 \leq p < \infty, s \geq 1$ and $(s,p) \notin \mathbb{N} \times \{1\}$. This fact was used in the proof of Theorem 5.3.

The proof of Theorem 5.4 rests on the following lemma:

Lemma 5.8 *There exists $C > 0$ such that for any $\omega \in C^\infty(\Lambda^l S^N)$, with $H(\omega) = 0$, we have:*

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)}.$$

Indeed, if this lemma is true, let $T \in W^{s-2,1}(\Lambda^l S^N)$ satisfying $H(T) = 0$. Then, there is a sequence of smooth $T_n \in C^\infty(\Lambda^l S^N)$ converging to T in $W^{s-2,1}(\Lambda^l S^N)$. Since H is continuous on $W^{s-2,1}$ (into a finite dimensional space), the sequence $H(T_n)$ converges to 0. Hence, we can assume that $H(T_n) = 0$ (by replacing T_n with $T_n - H(T_n)$).

For each n, there exists $\omega_n \in C^\infty(\Lambda^l S^N)$ such that $\Delta\omega_n = T_n$ and $H(\omega_n) = 0$ for every n. From Lemma 5.8 and the fact that $\Delta(\omega_p - \omega_q) = T_p - T_q$, it follows that

$$||\omega_p - \omega_q||_{W^{s,1}(\Lambda^l S^N)} \leq C||T_p - T_q||_{W^{s-2,1}(\Lambda^l S^N)}.$$

This shows that (ω_n) is a Cauchy sequence in $W^{s,1}(\Lambda^l S^N)$. So, it converges to some $\omega \in W^{s,1}(\Lambda^l S^N)$ which satisfies $\Delta\omega = T$ and the estimate

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C||T||_{W^{s-2,1}(\Lambda^l S^N)}$$

follows.

So it remains to prove Lemma 5.8. The proof relies on the following three lemmas:

Lemma 5.9 *There exists $C_0 > 0$ such that for any $w \in C_c^\infty(\mathbb{R}^N)$, we have:*

$$||w||_{W^{s,1}(\mathbb{R}^N)} \leq C_0(||w||_{W^{s-2,1}(\mathbb{R}^N)} + ||\Delta w||_{W^{s-2,1}(\mathbb{R}^N)}).$$

Proof: Thanks to the lifting property (see [78], Proposition 2.1.4.1), we have:

$$\begin{aligned} ||w||_{W^{s,1}(\mathbb{R}^N)} &\leq C||\mathcal{F}^{-1}(1+|y|^2)\mathcal{F}w||_{W^{s-2,1}(\mathbb{R}^N)} \\ &= C||(-\Delta + I)w||_{W^{s-2,1}(\mathbb{R}^N)} \\ &\leq C(||\Delta w||_{W^{s-2,1}(\mathbb{R}^N)} + ||w||_{W^{s-2,1}(\mathbb{R}^N)}). \end{aligned}$$

□

We proceed with the slightly more elaborate lemma, where we use the notation $I(l,N) := \{(i_1 < .. < i_l) : 1 \leq i_1 < .. < i_n \leq N\}$.

Lemma 5.10 *Let V be an open neighborhood of $0 \in \mathbb{R}^N$. Let $a^{IJ\alpha\beta} \in C^\infty(\bar{V})$ for any $I \in I(l,N), J \in I(l,N)$ and any $\alpha \in [|1,N|], \beta \in [|1,N|]$. We assume that $a^{IJ\alpha\beta}(0) = \delta_{IJ}\delta_{\alpha\beta}$. Then, there exists $\rho > 0, C > 0$ such that for any $\omega_J \in C_c^\infty(B(0,\rho)), J \in I(l,N)$, we have:*

$$||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} \leq C(||(T_J)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + ||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)})$$

where T_I denotes:

$$T_I := \sum_J \sum_{\alpha,\beta} a^{IJ\alpha\beta} \frac{\partial^2 \omega_J}{\partial x_\alpha \partial x_\beta}\ ,\ I \in I(l,N).$$

Here, the norm $||(\omega_J)||_{W^{s,1}(\mathbb{R}^N)}$ means (for instance)

$$||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} := \sum_J ||\omega_J||_{W^{s,1}(\mathbb{R}^N)}.$$

Proof of Lemma 5.10: Let us pick some $\rho > 0$ which will be subsequently subject to some restrictions (independent from the ω_J's). Let $\omega_J \in C_c^\infty(B(0,\rho))$, $J \in I(l,N)$. For any I, we have:

$$||\sum_\alpha \partial_{x_\alpha}\partial_{x_\alpha}\omega_I||_{W^{s-2,1}(\mathbb{R}^N)} = ||\sum_{J,\alpha,\beta} a^{IJ\alpha\beta}(0)\partial_{x_\alpha}\partial_{x_\beta}\omega_J||_{W^{s-2,1}(\mathbb{R}^N)}$$

$$\leq ||\sum_{J,\alpha,\beta} \partial_{x_\alpha}\partial_{x_\beta}((a^{IJ\alpha\beta}(0) - a^{IJ\alpha\beta})\omega_J)||_{W^{s-2,1}(\mathbb{R}^N)}$$

$$+||\sum_{J,\alpha,\beta} \partial_{x_\alpha}\partial_{x_\beta}(a^{IJ\alpha\beta}\omega_J)||_{W^{s-2,1}(\mathbb{R}^N)}$$

$$\leq || \sum_{J,\alpha,\beta}(a^{IJ\alpha\beta}(0)-a^{IJ\alpha\beta})\omega_J||_{W^{s,1}(\mathbb{R}^N)} + || \sum_{J,\alpha,\beta} a^{IJ\alpha\beta}\partial_{x_\alpha}\partial_{x_\beta}\omega_J||_{W^{s-2,1}(\mathbb{R}^N)}$$
$$+c||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)} =: a_1+a_2+a_3.$$

where c depends only on the $a^{IJ\alpha\beta}$'s.

To estimate the term a_1, we use Lemma 4.6.2.2 in [78] with ϕ being a function in $C_c^\infty(\mathbb{R}^N)$ equal to 1 on a neighborhood of $\bar{B}(0,1)$ and $\sigma := s-1$:

$$||[a^{IJ\alpha\beta}(.)-a^{IJ\alpha\beta}(0)]\omega_J||_{W^{s,1}(\mathbb{R}^N)} \leq c(\rho||\omega_J||_{W^{s,1}(\mathbb{R}^N)} + C_\rho||\omega_J||_{W^{\sigma,1}(\mathbb{R}^N)})$$

where c depends only on the $a^{IJ\alpha\beta}$'s. This implies that a_1 is not greater than

$$N^2c\rho||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} + N^2cC_\rho||(\omega_J)||_{W^{\sigma,1}(\Lambda^l\mathbb{R}^N)}.$$

The term a_2 is exactly $||T_I||_{W^{s-2,1}(\mathbb{R}^N)}$. Finally, we have shown that:

$$||\Delta\omega_I||_{W^{s-2,1}(\mathbb{R}^N)} \leq C||(T_J)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + C||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)}$$
$$+N^2c\rho||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)}.$$

This implies (thanks to Lemma 5.9) that:

$$||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} \leq C||(T_J)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + C||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)}$$
$$+N^3c\rho||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)}$$

and finally if we choose $\rho < 1/(2N^3c)$ (which depends only on the $a^{IJ\alpha\beta}$'s),

$$||(\omega_J)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} \leq C||(T_J)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + C||(\omega_J)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)}.$$

Lemma 5.10 is proved.

□

Lemma 5.11 *Let $x_0 \in S^N$. Then, there exists an open neighborhood U of x_0 and some constant $C>0$ such that for any $\omega \in C^\infty(\Lambda^l S^N)$ compactly supported in U we have*

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C(||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega||_{W^{s-1,1}(\Lambda^l S^N)}).$$

Proof of Lemma 5.11: The point x_0 belongs to the domain U_0 of a chart ϕ_0 such that $\phi_0(x_0)=0$ and $g_{ij}(x_0)=\delta_{ij}$. Let $V_0 := \phi(U_0)$. Let $\omega \in C_c^\infty(\Lambda^l U_0)$ and $T := \Delta\omega$. Then, for any $\eta \in C_c^\infty(\Lambda^l U_0)$, we have:

$$\langle d\omega, d\eta\rangle + \langle\delta\omega,\delta\eta\rangle = -\langle T,\eta\rangle$$

Let $\mu := \phi_{0\sharp}\omega =: \sum_I \mu_I e_I^*$ (where $e_I^* = e_{i_1}^* \wedge .. \wedge e_{i_l}^*$ and (e_i^*) is the dual basis of the canonical basis (e_i) of $\mathbb{R}^N$). Then, for each I, the μ_J's satisfy an equation of the form (see [71], chapter 7):

$$\sum_{J,\alpha,\beta} a^{IJ\alpha\beta}\partial_{x_\alpha}\partial_{x_\beta}\mu_J = T_I$$

on V_0, where T_I is a sum of terms involving $\phi_{0\sharp}T, \mu_J$ and the first derivatives of the μ_J's. Hence, the following estimate holds:

$$||T_I||_{W^{s-2,1}(\mathbb{R}^N)} \leq C(||T||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega||_{W^{s-1,1}(\Lambda^l S^N)}).$$

Thanks to Lemma 5.10 for these $a^{IJ\alpha\beta}$, (which satisfy $a^{IJ\alpha\beta}(0) = \delta_{IJ}\delta_{\alpha\beta}$, see [71], page 296), there exists $\rho > 0$ such that

$$||(\mu_I)||_{W^{s,1}(\Lambda^l\mathbb{R}^N)} \leq C(||(T_I)||_{W^{s-2,1}(\Lambda^l\mathbb{R}^N)} + ||(\mu_I)||_{W^{s-1,1}(\Lambda^l\mathbb{R}^N)})$$

if ω is compactly supported in $U := \phi_0^{-1}(B(0,\rho))$. This shows that

$$||\omega||_{W^{s,1}(\Lambda^l S^N)} \leq C(||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega||_{W^{s-1,1}(\Lambda^l S^N)}),$$

as required. Lemma 5.11 is proved.

□

We now complete the proof of Lemma 5.8. There exists a finite covering $U_1, .., U_r$ around some points $x_1, ..., x_r$ such that the previous lemma is true on each of these U_i. We introduce a partition of unity (ζ_i) corresponding to this covering. Now, let $\omega \in C^\infty(\Lambda^l S^N)$ and $\omega^j := \zeta_j\omega$. Thanks to Lemma 5.11, we have for every j :

$$\begin{aligned} ||\omega^j||_{W^{s,1}(\Lambda^l S^N)} &\leq C(||\Delta\omega^j||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega^j||_{W^{s-1,1}(\Lambda^l S^N)}) \\ &\leq C(||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)} + ||\omega||_{W^{s-1,1}(\Lambda^l S^N)}), \end{aligned} \tag{5.17}$$

thanks to the multiplication property.

Furthermore, the Green operator is continuous from $W^{s-2,1}(\Lambda^l S^N)$ into $W^{s-1,1}(\Lambda^l S^N)$. Indeed, the space $W^{s-2,1}(\Lambda^l S^N)$ is continuously embedded into $W^{-1,1+\epsilon}(\Lambda^l S^N)$ (say for $\epsilon := (s-1)/(N+1-s)$, see [78], Theorem 2.2.3). The Green operator is continuous from $W^{-1,1+\epsilon}(\Lambda^l S^N)$ into $W^{1,1+\epsilon}(\Lambda^l S^N)$ (thanks to Proposition 5.3), which is continuously embedded in $W^{s-1,1}(\Lambda^l S^N)$. This implies that for some constant C, we have:

$$||\omega||_{W^{s-1,1}(\Lambda^l S^N)} \leq C||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)}$$

(since, by hypothesis, $H(\omega) = 0$). Then, (5.17) implies

$$||\omega^j||_{W^{s,1}(\Lambda^l S^N)} \leq C||\Delta\omega||_{W^{s-2,1}(\Lambda^l S^N)}.$$

This completes the proof of Lemma 5.8.

Appendix A

Appendix to Chapter 1: Some further results on the Lower Bounded Slope Condition

In this appendix, we give some further results on the lower bounded slope condition. As in Chapter 1, we consider a bounded convex open subset Ω in $\mathbb{R}^n$ and a map

$$\phi : \Gamma \to \mathbb{R},$$

where Γ is the boundary of Ω. Throughout this appendix, we will assume that ϕ is lower semicontinuous.

A.1 A weaker lower bounded slope condition

In light of the definition of the lower bounded slope condition, it is natural to introduce:

Definition A.1 *Let $x \in \Gamma$. Then we define the convex subdifferential of ϕ at x by*

$$\partial\phi(x) := \{\zeta \in \mathbb{R}^n : \forall y \in \Gamma,\ \phi(y) \geq \phi(x) + \langle \zeta, y - x \rangle\}.$$

As usual, an element of the convex subdifferential is called a convex *subgradient*. A map ϕ satisfies the lower bounded slope condition if and only if there exists $Q \geq 0$ such that for any $x \in \Gamma$:

$$\partial\phi(x) \cap B(0, Q) \neq \emptyset.$$

In chapter 1, we have already defined the *proximal* subdifferential of ϕ at $x \in \Gamma$. We recall that $\zeta \in \partial_P \phi(x)$ if and only if $\zeta \in \partial_P \tilde{\phi}(x)$ where $\tilde{\phi}(x)$ is defined by

$$\tilde{\phi} : x \in \mathbb{R}^n \mapsto \begin{cases} \phi(x) & \text{when } x \in \Gamma, \\ +\infty & \text{when } x \notin \Gamma. \end{cases}$$

It is obvious that $\partial\phi(x) \subset \partial_P\phi(x)$. We will say that ϕ satisfies a *weak lower bounded slope condition* (or that ϕ *is WLBSC* for short) if

$$\forall x \in \Gamma, \partial\phi(x) \neq \emptyset.$$

(When ϕ satisfies the lower bounded slope condition, we will say that ϕ *is LBSC*).

Note that if ϕ is $WLBSC$, then it is bounded from below (since $\phi(y) \geq \phi(x)+ \langle \zeta, y-x\rangle$ for any $y, x \in \Gamma$, $\zeta \in \partial\phi(x)$).

From the properties of $\partial_P\tilde{\phi}(x)$, we get:

Proposition A.1 *i) For any $\zeta \in \mathbb{R}^n$, $\zeta \in \partial_P\phi(x)$ if and only if there exists $g \in C^2(\mathbb{R}^n, \mathbb{R})$ such that $\phi - g|_\Gamma$ has a local minimum on Γ at x and $\nabla g(x) = \zeta$.*

ii) For any $\zeta \in \mathbb{R}^n$, $\zeta \in \partial_P\phi(x)$ if and only if

$$\liminf_{y \to x, y \neq x} \frac{\phi(y) - \phi(x) - \langle \zeta, y - x\rangle}{|y-x|^2} > -\infty.$$

In particular, if ϕ is the restriction of a C^2 map defined on a neighborhood of some $x \in \Gamma$, then $\partial_P\phi(x) \neq \emptyset$.

By definition of the convex subdifferential and the proximal subdifferential, we have $\partial\phi(x) \subset \partial_P\phi(x)$. Conversely, when Ω is uniformly convex (see Definition 2 in Chapter 1), we have $\partial_P\phi(x) \subset \partial\phi(x)$. (This is an easy consequence of the definitions, and we omit the proof). The last statement gives a convenient way to show that a map ϕ is $WLBSC$ in case when Ω is uniformly convex.

It is natural to ask whether a function which is $WLBSC$ is necessarily $LBSC$ and if not, does there exist a further condition which guarantees this implication? Note that if we had defined these conditions $WLBSC$ and $LBSC$ for maps defined on an open subset of $\mathbb{R}^n$ (instead of Γ) in a similar way, then the answer to this question would be easy: it amounts to the fact that a convex function is locally Lipschitz. The answer is quite different in our situation: it depends on the geometry of Ω.

For instance, we have:

Proposition A.2 *Assume that $n = 2$ and that Ω is a square:*

$$\Omega := \{(x, y) \in \mathbb{R}^2 : \max(|x|, |y|) < 1\}.$$

Let $\phi : \Gamma \to \mathbb{R}$ be $WLBSC$. Then ϕ is $LBSC$.

Proof: Consider the map

$$g : x \in [-1, 1] \mapsto \phi(x, 1).$$

It is convex on $[-1, 1]$ and the convex subdifferential of g at -1 and 1 is not empty. Hence, g is Lipschitz on $[-1, 1]$ and there exists Q such that $\partial g(x) \subset [-Q, Q]$ for any $x \in [-Q, Q]$. Repeating this on each side of the square and by increasing Q if necessary, we may assume that ϕ is Lipschitz of rank Q on Γ.

We denote by $\zeta(x) \in \mathbb{R}$ an (arbitrary) subgradient of g at $x \in [-1, 1]$. We claim that for L sufficiently large, we have $(\zeta(\bar{x}), L) \in \partial\phi(\bar{x}, 1)$ for any $\bar{x} \in [-1, 1]$. To prove this, we proceed to verify the inequality

$$\phi(x, y) \geq \phi(\bar{x}, 1) + \zeta(\bar{x})(x - \bar{x}) + L(y - 1) \tag{A.1}$$

on each side of the square.

On $[-1,1]\times\{1\}$, (A.1) is obvious (for any $L\geq 0$) since $y=1$ (here, we also use the definition of $\zeta(\bar{x})$).

On $[-1,1]\times\{-1\}$,

$$\phi(x,-1)\geq\phi(x,1)-|\phi(x,-1)-\phi(x,1)|\geq\phi(\bar{x},1)+\zeta(\bar{x})(x-\bar{x})+L(-1-1)$$

provided that $2L\geq|\phi(x,-1)-\phi(x,1)|$. This will be certainly the case if

$$L\geq||\phi||_{L^\infty(\Gamma)}.$$

For these values of L, (A.1) is true when $(x,y)\in[-1,1]\times\{-1\}$.

On $\{-1\}\times[-1,1]$, we use the fact that

$$\phi(-1,y)\geq\phi(-1,1)-Q|y-1|\geq\phi(\bar{x},1)+\zeta(\bar{x})(-1-\bar{x})+L(y-1)$$

provided that $L\geq Q$. The case $\{1\}\times[-1,1]$ is very similar and we omit it.

Finally, (A.1) is true on each side of the square for

$$L:=\max(Q,||\phi||_{L^\infty(\Gamma)}).$$

This shows that for any $\bar{x}\in[-1,1]$,

$$\partial\phi(\bar{x},1)\cap\bar{B}(0,(Q^2+L^2)^{1/2})\neq\emptyset.$$

The same can be done for any $(\bar{x},\bar{y})\in\Gamma$. This shows that ϕ satisfies the lower bounded slope condition.

□

The same proof shows that if ϕ is convex on each side of the square and globally Lispchitz, then it satisfies the lower bounded slope condition. However, a map which is $WLBSC$ is not necesaarily globally Lipschitz. It may happen that it is even not bounded. Consider the following examples.

The set Ω is the unit disc in $\mathbb{R}^2$ so that each point of Γ can be written as $(\cos\theta,\sin\theta)$ with $\theta\in[0,2\pi[$. Then, the map

$$\phi:(\cos\theta,\sin\theta)\in\Gamma\mapsto\frac{1}{2\pi-\theta}$$

is not bounded and yet, ϕ is $WLBSC$. To see this, it is enough to show that $\partial_P\phi(\cos\theta,\sin\theta)\neq\emptyset$ for any $\theta\in[0,2\pi)$ (since Ω is uniformly convex). This fact is obvious when $\theta\in(0,2\pi)$ since in this case, ϕ is the restriction of a C^2 map defined on a neighborhood of $(\cos\theta,\sin\theta)$. When $\theta=0$, we can easily check that $(0,0)\in\partial\phi(1,0)$ (this is a consequence of the fact that ϕ has a global minimum at $(1,0)$). Hence, ϕ is $WLBSC$ but not bounded.

The map

$$\phi:(\cos\theta,\sin\theta)\in\Gamma\mapsto\theta$$

is $WLBSC$ and bounded but not continuous.

The map

$$\phi:(\cos\theta,\sin\theta)\in\Gamma\mapsto\sqrt{|\sin\theta|}$$

is $WLBSC$ and continuous but not Lipschitz.

The following theorem gives a necessary and sufficient condition for a map ϕ which is $WLBSC$ to be $LBSC$.

Theorem A.1 *Assume that ϕ is WLBSC. For any $x \in \Gamma$ and any $\zeta \in \partial\phi(x)$, we denote*

$$\pi_{x,\zeta}(y) := \phi(x) + \langle \zeta, y - x \rangle \quad , \ \forall y \in \mathbb{R}^n.$$

Then ϕ is LBSC if and only if for any $x_* \in \Omega$

$$I := \inf_{x \in \Gamma} \sup_{\zeta \in \partial\phi(x)} \pi_{x,\zeta}(x_*) > -\infty. \tag{A.2}$$

Proof: Assume first that ϕ is *LBSC*. Then, for any $x \in \Gamma$, there exists $\zeta_x \in \partial\phi(x) \cap \bar{B}(0, Q)$. Whence

$$\sup_{\zeta \in \partial\phi(x)} \pi_{x,\zeta}(x_*) \geq -Q \operatorname{diam} \Omega + \inf_{\Gamma} \phi$$

for any $x_* \in \Omega$. This completes the proof of $I > -\infty$.

Conversely, fix $x_* \in \Omega$ and assume that there exists $M \in \mathbb{R}$ such that $I \geq M$. Then, for any $x \in \Gamma$, there exists $\zeta_x \in \partial\phi(x)$ such that

$$\pi_{x,\zeta_x}(x_*) \geq M - 1.$$

Recall that for any $n \neq 0$ in the convex normal cone to Ω at $x \in \Gamma$, we have

$$\langle n, y - x \rangle \leq 0 \quad , \ \forall y \in \bar{\Omega}.$$

Hence, $\zeta_x + n \in \partial\phi(x)$ and by replacing n by tn for some $t > 0$ if necessary, we may assume that $\pi_{x,\zeta_x+n}(x_*) = M-1$ (here, we use the fact that $\langle n, x_* - x \rangle < 0$). We denote $\zeta_x + n$ by ξ_x. Now, we set

$$\rho : y \in \mathbb{R}^n \mapsto \sup_{x \in \Gamma} (\pi_{x,\xi_x}(y)).$$

The map ρ satisfies the following properties:
i) ρ is convex,
ii) ϕ is the restriction of ρ to Γ,
iii) $\rho(x_*) = M - 1$,
iv) $\forall x \in \Gamma$, $\forall \theta \geq 0$,

$$\rho((1-\theta)x_* + \theta x) = (1-\theta)\rho(x_*) + \theta\rho(x).$$

Property iv) implies that ρ is finite everywhere on $\mathbb{R}^n$. This shows that ϕ is the restriction to Γ of a finite convex function ρ. By Proposition 1 in Chapter 1, ϕ satisfies the lower bounded slope condition.

□

A.2 A version of a theorem of Hartman for the lower bounded slope condition

We recall first one of the main theorems in [43]. We fix some $x_* \in \Omega$. For any three points in Γ, we say that (x_0, x_{01}, x_1) is *well ordered* if $x_1 - x_*$ and $x_2 - x_*$ are not proportional and if

$$\exists (\lambda, \mu) \in \mathbb{R}^{*+} \mid x_{01} - x_* = \lambda(x_0 - x_*) + \mu(x_1 - x_*).$$

An easy consequence of the definition is that (x_0, x_{01}, x_1) is well ordered if and only if (x_1, x_{01}, x_0) is well ordered.

Let (x_0, x_{01}, x_1) be well ordered. Then the family $\{(x_0 - x_*, x_1 - x_*)\}$ gives an orientation to the 2 plane $\Pi := \text{Vect}\,(x_0 - x_*, x_1 - x_*)$. We will say that a system of coordinates (with x_* as origin) on Π is *well oriented* (relatively to (x_0, x_{01}, x_1)) if it is defined by an orthonormal basis which has the same orientation as the basis $\{(x_0 - x_*, x_1 - x_*)\}$. We can now state (see [43], Corollary 2.1)

Theorem A.2 *The map ϕ satisfies the lower bounded slope condition if and only if there exists $N \geq 0$ such that for any $z_* \leq -N$,*

$$\begin{vmatrix} \xi_0 & \eta_0 & \phi(x_0) - z^* \\ \xi_{01} & \eta_{01} & \phi(x_{01}) - z^* \\ \xi_1 & \eta_1 & \phi(x_1) - z^* \end{vmatrix} \geq 0 \tag{A.3}$$

for any well ordered (x_0, x_{01}, x_1). We have denoted by (ξ_0, η_0), (ξ_{01}, η_{01}) and (ξ_1, η_1) respectively the coordinates of x_0, x_{01}, x_1 in any well oriented system of coordinates on $\Pi = \mathit{Vect}\,(x_0 - x_*, x_1 - x_*)$.

Proof: For any $z_* \in \mathbb{R}$, we can define

$$\tilde{\rho}((1-t)x_* + tx) := (1-t)z_* + t\phi(x) \quad \forall x \in \Gamma, \quad \forall t \geq 0. \tag{A.4}$$

Note first that ϕ satisfies the lower bounded slope condition if and only if $\tilde{\rho}$ is convex for any $|z_*|$ sufficiently large. Indeed, if $\tilde{\rho}$ is convex, then ϕ is the restriction of the convex function $\tilde{\rho}$ to Γ. This implies that ϕ satisfies the lower bounded slope condition. Conversely, if ϕ satisfies the lower bounded slope condition, then the proof of Theorem A.1 shows that for any $|M|$ sufficiently large, we can define a convex map ρ which satisfies the four properties i), ii), iii), iv). If we choose $z_* = M - 1$ in (A.4) then $\tilde{\rho} = \rho$ (since $\tilde{\rho} = \rho$ on x_*, on Γ and is affine on each half line $\{x_* + t(x - x_*) : t \geq 0, x \in \Gamma\}$). This implies that $\tilde{\rho}$ is convex for $|z_*|$ sufficiently large.

We proceed to show that (A.3) is equivalent to the convexity of $\tilde{\rho}$. The convexity of $\tilde{\rho}$ is equivalent to the convexity of $\tilde{\rho}$ restricted to any line L. If $x_* \in L$, then the convexity of $\tilde{\rho}|_L$ is obvious. Consider now a line L such that $x_* \notin L$. Then, we may choose a system of coordinates (ξ, η) in the 2 plane generated by x_* and L such that $x_* = (0, 0)$ and $L := \{(c, t) : t \in \mathbb{R}\}$ for some $c > 0$.

So, $\tilde{\rho}|_L$ is convex if and only if for any $y_0 := (c, \alpha_0)$, $y_1 := (c, \alpha_1) \in L$, with $\alpha_0 < \alpha_1$ and any $\theta \in (0, 1)$, we have

$$\tilde{\rho}(y_{01}) \leq \theta\tilde{\rho}(y_0) + (1-\theta)\tilde{\rho}(y_1), \tag{A.5}$$

where $y_{01} = (c, \theta\alpha_0 + (1-\theta)\alpha_1)$. Let us denote by $x_i = (\xi_i, \eta_i)$ $(i = 0, 1, 01)$ the intersections of Γ with the half-line $\{x_* + t(y_i - x_*) : t \geq 0\}$. Then, (x_0, x_{01}, x_1) is well ordered and the system of coordinates (ξ, η) is well oriented (relatively to (x_0, x_{01}, x_1)).

Let us denote by $t_0, t_{01}, t_1 > 0$ the numbers which satisfy $y_j = x_* + t_j(x_j - x_*)$, $j = 0, 01, 1$. Then,

$$\tilde{\rho}(y_j) = z_* + t_j(\phi(x_j) - z_*)\ ,\ t_j = \frac{c}{\xi_j}.$$

Hence, (A.5) is equivalent to

$$\frac{\phi(x_{01})-z_*}{\xi_{01}} \leq \theta\frac{\phi(x_0)-z_*}{\xi_0} + (1-\theta)\frac{\phi(x_1)-z_*}{\xi_1}.$$

Since (x_0, x_{01}, x_1) is well ordered, we have $\xi_0\eta_1 - \xi_1\eta_0 > 0$. Using this in the inequality above, we get

$$(\xi_0\eta_1 - \xi_1\eta_0)(\phi(x_{01}) - z_*) \leq (\xi_{01}\eta_1 - \xi_1\eta_{01})(\phi(x_0) - z_*) +$$
$$(\xi_0\eta_{01} - \xi_{01}\eta_0)(\phi(x_1) - z_*)$$

which is equivalent to (A.3). This completes the proof of Theorem A.2.

□

Note that (A.3) is equivalent to

$$\inf \frac{\begin{vmatrix} \xi_0 & \eta_0 & \phi(x_0) \\ \xi_{01} & \eta_{01} & \phi(x_{01}) \\ \xi_1 & \eta_1 & \phi(x_1) \end{vmatrix}}{\begin{vmatrix} \xi_0 & \eta_0 & 1 \\ \xi_{01} & \eta_{01} & 1 \\ \xi_1 & \eta_1 & 1 \end{vmatrix}} > -\infty \quad ,$$

where the infimum is taken over all the well ordered $(x_0, x_{01}, x_1) \in \Gamma^3$ which are not on the same line (and (ξ_0, η_0), (ξ_{01}, η_{01}), (ξ_1, η_1) denote their coordinates in a well oriented system of coordinates). Note also that

$$\begin{vmatrix} \xi_0 & \eta_0 & 1 \\ \xi_{01} & \eta_{01} & 1 \\ \xi_1 & \eta_1 & 1 \end{vmatrix}$$

is the area $\mathcal{A}_{x_0,x_{01},x_1}$ of the 2 simplex with vertices x_0, x_{01}, x_1. Similarly,

$$\frac{\begin{vmatrix} \xi_0 & \eta_0 & \phi(x_0) \\ \xi_{01} & \eta_{01} & \phi(x_{01}) \\ \xi_1 & \eta_1 & \phi(x_1) \end{vmatrix}}{\begin{vmatrix} \xi_0 & \eta_0 & 1 \\ \xi_{01} & \eta_{01} & 1 \\ \xi_1 & \eta_1 & 1 \end{vmatrix}} = \frac{\phi(x_1)\mathcal{A}_{x_0,x_{01},x_*} - \phi(x_{01})\mathcal{A}_{x_0,x_*,x_1} + \phi(x_0)\mathcal{A}_{x_*,x_{01},x_1}}{\mathcal{A}_{x_0,x_{01},x_1}}. \tag{A.6}$$

This quantity does not involve the derivatives of ϕ but only the values of ϕ. It is reminiscent of a property of convex functions defined on an open subset U of $\mathbb{R}^n$, namely: $f : U \to \mathbb{R}$ is convex if and only if

$$\liminf_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{|h|^2} \geq 0,$$

(to be compared with (A.6)).

A.3 Density results

In this section, we show how the result of [45] can be generalized to the setting of maps satisfying the lower bounded slope condition. Recall that a map which satisfies the lower bounded slope condition is convex on any line contained in Γ. Let $\Lambda(\Gamma)$ be the set of continuous maps $\phi : \Gamma \to \mathbb{R}$ which are convex on any line contained in Γ. (In particular, when Ω is strictly convex, $\Lambda(\Gamma) = C^0(\Gamma, \mathbb{R})$.) Then we have:

Theorem A.3 *The set of maps $\phi : \Gamma \to \mathbb{R}$ which satisfy the lower bounded slope condition is dense in $\Lambda(\Gamma)$ (for the topology of $C^0(\Gamma, \mathbb{R})$).*

Proof: (This is a mere adaptation of the proof in [45]). Let $\phi \in \Lambda(\Gamma)$. Without loss of generality, we may assume that $0 \in \Omega$. Let $r > ||\phi||_{L^\infty(\Gamma)}$ and

$$\phi_r(x) := \inf_{S(x)} \{-r + T \sum_{i=1}^{m} \mu_i(\phi(x_i) + r)\} \ , \ x \in \mathbb{R}^n,$$

with

$$S(x) := \{(T, \mu_1, .., \mu_m, x_1, .., x_m) : T \sum_{i=1}^{m} \mu_i x_i = x; T \geq 0,$$

$$\mu_i \geq 0, \sum \mu_i = 1, x_i \in \Gamma, m > 0\}.$$

Since $\phi(x_i) + r \geq 0$, we have $\phi_r(x) \geq -r$. Using the fact that $(T = 1, \mu = 1, x) \in S(x)$, we get $\phi_r \leq \phi$. When $x \notin \Omega$, $T \geq 1$. Since

$$\phi_r(x) := \inf_{S(x)} \{(T - 1)r + T \sum_{i=1}^{m} \mu_i \phi(x_i)\} \ , \ x \in \mathbb{R}^n,$$

we have $\phi_r(x) \geq \phi_s(x)$ when $r \geq s$.

We claim that ϕ_r is convex on $\mathbb{R}^n$. Indeed, let $x, x' \in \mathbb{R}^n, \theta \in (0, 1)$ and

$$(T, \mu_1, .., \mu_m, x_1, .., x_m) \in S(x) \ , \ (T', \mu'_1, .., \mu'_{m'}, x'_1, .., x'_{m'}) \in S(x').$$

Then,

$$(\theta T + (1 - \theta)T', \frac{\theta T}{\theta T + (1 - \theta)T'}\mu_1, .., \frac{\theta T}{\theta T + (1 - \theta)T'}\mu_m,$$

$$\frac{(1 - \theta)T}{\theta T + (1 - \theta)T'}\mu'_1, .., \frac{(1 - \theta)T}{\theta T + (1 - \theta)T'}\mu'_{m'}, x_1, ..x_m, x'_1, .., x'_{m'})$$

$$\in S(\theta x + (1 - \theta)x').$$

Hence

$$\phi_r(\theta x + (1 - \theta)x') \leq -r + (\theta T + (1 - \theta)T')[\sum_{i=1}^{m} \frac{\theta T}{\theta T + (1 - \theta)T'}\mu_i(\phi(x_i) + r)$$

$$+ \sum_{i=1}^{m'} \frac{(1 - \theta)T'}{\theta T + (1 - \theta)T'}\mu'_i(\phi(x'_i) + r)]$$

$$\leq \theta(-r + T \sum_{i=1}^{m} \mu_i(\phi(x_i) + r)) + (1 - \theta)(-r + T' \sum_{i=1}^{m'} \mu'_i(\phi(x'_i) + r)),$$

and finally

$$\phi_r(\theta x + (1 - \theta)x') \leq \theta\phi_r(x) + (1 - \theta)\phi_r(x').$$

This shows that ϕ_r is convex and completes the proof of the claim.

Assume now that $r > 2||\phi||_{L^\infty(\Gamma)}+1$. Let $x \in \Gamma$. Then there exists a sequence

$$(T^k, \mu_1^k, .., \mu_{m^k}^k, x_1^k, .., x_{m^k}^k) \in S(x)$$

such that

$$-r + T^k \sum_{i=1}^{m^k} \mu_i^k (\phi(x_i^k) + r) - \phi_r(x) \leq 1/k.$$

Since $x \notin \Omega$, $T^k \geq 1$. Moreover, $\phi_r(x) \leq \phi(x) \leq M$, where $M := ||\phi||_{L^\infty(\Gamma)}$. Hence,

$$(T^k - 1)r - T^k M \leq 1/k + M$$

so that

$$1 \leq T^k \leq \frac{1}{1-(2M+1)/r}.$$

Let us denote by μ_k the probability measure defined on Γ by

$$\mu^k := \sum_{i=1}^{m^k} \mu_i^k \delta_{x_i^k}$$

where δ_x is the Dirac measure at x. Then,

$$x = T^k \int_\Gamma y \, d\mu^k \quad \text{and} \quad \sum_{i=1}^{m^k} \mu_i^k \phi(x_i^k) = \int_\Gamma \phi \, d\mu^k.$$

Passing to a subsequence, we may assume that T^k converges to some

$$T \in [1, \frac{1}{1-(2M+1)/r}]$$

while μ^k weakly converges to some μ such that

$$x = T \int_\Gamma u \, d\mu \quad \text{and} \quad \phi_r(x) = -r + T \int_\Gamma (\phi(y) + r) \, d\mu^k. \tag{A.7}$$

This expression of ϕ_r will be useful to complete the proof of the theorem.

Since the restriction of ϕ_r to Γ satisfies the lower bounded slope condition (this is a consequence of the fact that ϕ_r is convex), it remains to prove that $\phi_r|_\Gamma$ converges uniformly to ϕ.

Let $\sigma_r := \max_{x\in\Gamma}(\phi(x) - \phi_r(x))$. Then $(\sigma_r)_r$ is nonnegative and nonincreasing so it converges to some $c \geq 0$. Furthermore, there exists a sequence of points $x_j \in \Gamma$ such that $c \leq \sigma_j = \phi(x_j) - \phi_j(x_j)$.

Using (A.7) with $r = j, x = x_j$, we get

$$x_j = T_j \int_\Gamma y \, d\mu_j \ , \ \phi_j(x) = (T_j - 1)j + T_j \int_\Gamma \phi \, d\mu_j$$

for some $1 \leq T_j \leq \frac{1}{1-(2M+1)/j}$ and some probability measure μ on Γ.

Then, for j sufficiently large, we have

$$\phi(x_j) - c \geq \phi_j(x_j) \geq T_j \int_\Gamma \phi \, d\mu_j.$$

Passing to a subsequence, we may assume that x_j converges to some x_0 and that μ_j weakly converges to some probability measure μ on Γ. We then get

$$x_0 = \int_\Gamma y\, d\mu \ , \ \phi(x_0) - c \geq \int_\Gamma \phi(y)\, d\mu. \tag{A.8}$$

If x_0 is extremal in Γ, then the support of μ is $\{x_0\}$ and then $\phi(x_0) - c \geq \phi(x_0)$ so that $c = 0$. If x_0 is not extremal, let Δ be the largest flat piece of Γ containing x_0 in its interior (it is a closed convex subset of Γ). Then the support of μ is contained in Δ and

$$\phi(x_0) - c \geq \int_\Delta \phi\, d\mu \geq \phi(x_0)$$

so that $c = 0$. In any case, $c = 0$, which completes the proof of the theorem.

□

Using Theorem A.3, we get a slight generalization of Theorem 5 in Chapter 1: let

$$I : u \in W_0^{1,1}(\Omega) \mapsto \int_\Omega F(\nabla u(x))\, dx$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is strictly convex. We also assume that F is coercive: there exists $\alpha > 1$ and $a > 0, b \in \mathbb{R}$ such that

$$\forall p \in \mathbb{R}^n, \quad F(p) \geq a|p|^\alpha + b.$$

Finally, let $\phi : \Gamma \to \mathbb{R}$ be a continuous map which belongs to $\Lambda(\Gamma)$. We also assume that ϕ is the trace of a map in $W^{1,1}(\Omega)$. Then

Theorem A.4 *There exists one and only one minimum to I in $W_0^{1,\alpha}(\Omega) + \phi$.*

Proof: The direct method in the Calculus of variations shows that there exists a minimum u of I on $W_0^{1,\alpha}(\Omega) + \phi$. Moreover, this minimum is unique since I is strictly convex.

By Theorem A.3, there exists a sequence of maps $\phi_j : \Gamma \to \mathbb{R}$ satisfying the lower bounded slope condition, which converges uniformly to ϕ on Γ. For any j, there exists $u_j \in W_0^{1,\alpha}(\Omega) + \phi_j$ which minimizes I. Then, Theorem 1.2 in [28] shows that u_j is continuous on Ω. Moreover, the maximum principle (Corollary 4.1 in [62]) shows that

$$||u_j - u||_{L^\infty(\Omega)} \leq ||\phi_j - \phi||_{L^\infty(\Gamma)},$$

which shows that u_j converges uniformly to u on Ω. Hence, u is continuous on Ω.

□

There is a corresponding version of the theorem in which $F(p)$ is replaced by $F(p) + G(x, u)$ (for this case, one has to use the results in [13] instead of Theorem 1.2 in [28]).

Appendix B

Appendix to Chapter 2: Proof of Theorem 2.3

In this appendix, we give the proof of Chapter 2, Theorem 2.3.

B.1 Proof of Theorem 2.3

The hypothesis (HF) implies that for any ζ in the convex subdifferential of F at 0, we have $F(p) \geq F(0) + \langle \zeta, p \rangle + (\mu/2)|p|^2$, which gives:

$$F(p) \geq F(0) - |\zeta||p| + \frac{\mu}{2}|p|^2 \geq (\frac{\mu}{2} - \epsilon_0)|p|^2 - N \tag{B.1}$$

with $N := -F(0) + |\zeta|^2/4\epsilon_0$. We have used the inequality: $|\zeta||p| \leq \epsilon_0|p|^2 + |\zeta|^2/(4\epsilon_0)$. We fix from now on ϵ_0 such that $\mu_1 := \mu/2 - \epsilon_0 > q/\Lambda$.

Then, thanks to $(B.1)$ and the lower boundedness of G

$$G(x, u) \geq -q|u|^2 - Q(x)|u|^\delta - R(x), \tag{B.2}$$

the functional I is well defined on $W_0^{1,2}(\Omega) + \phi$.

In the hypothesis defining the local Lipschitzness of G, namely

$$|G(x, u) - G(x, u')| \leq M|u - u'|(1 + |u|^\beta + |u'|^\beta) \tag{B.3}$$

where β was supposed to be in $(0, 2^* - 1)$, we can actually assume that $\beta \geq 1$. If it were not the case, then we could use the fact that $|u|^\beta \leq 1 + |u|$, so that we would have:

$$|G(x, u) - G(x, u')| \leq (M + 2)|u - u'|(1 + |u| + |u'|).$$

According to this remark, we will assume in the following that $\beta \geq 1$.

To prove Theorem 2.3, we first need the following existence theorem:

Theorem B.1 *There exists a minimum to problem (P) on $W_0^{1,2}(\Omega) + \phi$. Moreover, any minimizing function is bounded in $L^\infty(\Omega)$ and in $W^{1,2}(\Omega)$ by a constant T which depends on Ω, ϕ, F and G.*

This theorem amounts to Theorem 8.1 in [10]. Nevertheless, it is important to analyse the dependence of the bound on $||u||_{L^\infty(\Omega)}$ with the data of the problem. That is why we give a proof of this theorem at the end of this appendix. Actually, Ω, F, ϕ will remain the same data for all the problems that we will consider in the following and the dependence on G appears through the dependence on $M, \beta, \delta, t, q, ||R||_{L^1}, ||Q||_{L^t}$. In fact, β, t, δ, q will denote the same constants throughout this note. The letter C will denote different constants depending on $\Omega, \phi, F, \delta, t, \beta, q$. On the contrary, the dependence of T on $||R||_{L^1}, ||Q||_{L^t}, M$ is of interest. The proof of Theorem B.1 will show that this dependence is polynomial so that T can be uniformly bounded provided that $||R||_{L^1}, ||Q||_{L^t}, M$ remain bounded.

We now prove Theorem 2.3 admitting Theorem B.1 . Let us denote by u_0 any solution to problem (P) on $W_0^{1,2}(\Omega) + \phi$. Throughout this note, S_p denotes the Sobolev constant in $W_0^{1,p}(\Omega)$.

Stability of the hypotheses on G under convolution. Let $\rho \in C_c^\infty(]-1,1[), 0 \leq \rho \leq 1, \int_{\mathbb{R}} \rho = 1$ and $\rho_\epsilon := 1/\epsilon\rho(\cdot/\epsilon)$. We also assume that ρ is even. Now, let

$$G_\epsilon(x,u) := \int_{\mathbb{R}} G(x, u - v)\rho_\epsilon(v)\, dv = \int_{]-1,1[} G(x, u - \epsilon v)\rho(v)\, dv.$$

Then, for any $u \in \mathbb{R}^n, G_\epsilon(\cdot, u)$ is still measurable. Indeed, if ρ were a finite sum of Dirac masses $\sum_{i=1}^r a_i\delta_{v_i}, a_i \geq 0, \sum_{i=1}^r a_i = 1$, then we would have:

$$G_\epsilon(x,u) = \sum_{i=1}^r a_i G(x, u - \epsilon v_i),$$

which is a measurable function, as a sum of measurable functions. Now, convex combinations of Dirac masses are dense in the closed unit ball of $(C([-1,1]))^*$ for the weak* topology (thanks to Krein Milmann Theorem for instance), so that there exists a sequence ρ_l of convex combinations of Dirac masses which converge in this topology to $\rho\, dv$. Since for almost everywhere $x, G(x, \cdot)$ is continuous, for such $x, \int_{[-1,1]} G(x, u - \epsilon v)\, d\rho_l(v)$ converges to $\int_{[-1,1]} G(x, u - \epsilon v)\rho(v)\, dv$. This shows that $G_\epsilon(\cdot, u)$ is measurable as the limit almost everywhere of measurable functions.

The hypothesis (B.2) becomes:

$$\begin{aligned}
G_\epsilon(x,u) &\geq \int_{]-1,1[} \{-q|u-\epsilon v|^2 - Q(x)|u - \epsilon v|^\delta - R(x)\}\rho(v)\, dv \\
&\geq -q|u|^2 - q\epsilon^2 \int_{]-1,1[} v^2\rho(v) + 2\epsilon uq \int_{]-1,1[} v\rho(v) \\
&\quad -(1+\epsilon)^\delta Q(x)|u|^\delta - (1 + 1/\epsilon)^\delta \epsilon^\delta \int_{]-1,1[} |v|^\delta \rho(v)\, dv - R(x)
\end{aligned}$$

Here, we have used the fact that $|u - \epsilon v|^\delta \leq (1+\epsilon)^\delta|u|^\delta + (1+1/\epsilon)^\delta|\epsilon v|^\delta$. Now, we use the fact that $|u| \leq 1 + |u|^2$, so that G_ϵ satisfies (B.2) with

$$q_\epsilon := q + 2\epsilon q \int_{]-1,1[} v\rho(v) \quad , \quad \delta_\epsilon := \delta \quad , \quad Q_\epsilon := (1+\epsilon)^\delta Q$$

$$R_\epsilon(x) := q\epsilon^2 - 2q\epsilon \int_{]-1,1[} v\rho(v) + (1+\epsilon)^\delta + R(x).$$

We have used that fact that $\int_{]-1,1[} |v|^\delta \rho(v)\, dv \leq 1$. Since ρ is even, q_ϵ reduces to q and R_ϵ reduces to

$$q\epsilon^2 + (1+\epsilon)^\delta + R(x).$$

The hypothesis (B.3) remains true (in the following, we use the fact that $|u - \epsilon v|^\beta \leq (1+\epsilon)^\beta(|u|^\beta + |v|^\beta)$):

$$\begin{aligned} |G_\epsilon(x,u) - G_\epsilon(x,u')| &\leq M \int_{-1}^{1} |u-u'|(1 + |u-\epsilon v|^\beta + |u' - \epsilon v|^\beta)\rho(v)\, dv \\ &\leq M_\epsilon |u-u'|(1+|u|^\beta + |u'|^\beta). \end{aligned}$$

Then (B.3) holds with $M_\epsilon := 3(1+\epsilon)^\beta M$.

Finally,

$$G_\epsilon(x, \bar{u}(x)) \leq G(x, \bar{u}(x)) + \epsilon M \int_{]-1,1[} \rho(v)(1 + |\bar{u}(x) - \epsilon v|^\beta + |\bar{u}(x)|^\beta)$$

$$\leq G(x,\bar{u}(x)) + \epsilon M |\bar{u}(x)|^\beta[1 + (1+\epsilon)^\beta] + \epsilon M(1 + (1+\epsilon)^\beta)$$

This shows that $\int_\Omega G_\epsilon(x,\bar{u})$ is finite, since $\bar{u} \in W^{1,2}(\Omega) \subset L^\beta(\Omega)$.

Stability of the hypotheses on G under penalization. Denote by

$$G_1(x,u) := G(x,u) + |u - u_0(x)|^2.$$

Then G_1 still satisfies (B.2) since $|u - u_0(x)|^2 \geq 0$. Moreover, $\int_{\mathbb{R}} G_1(x,\bar{u}) < \infty$. Finally,

$$\begin{aligned} |G_1(x,u) - G_1(x,u')| &\leq |G(x,u) - G(x,u')| + ||u-u_0(x)|^2 - |u' - u_0(x)|^2| \\ &\leq |G(x,u) - G(x,u')| + |u^2 - u'^2| + 2|u_0(x)||u-u'| \\ &\leq M|u-u'|(1+|u|^\beta + |u'|^\beta) \\ &\qquad + |u-u'|(|u| + |u'| + 2||u_0||_{L^\infty(\Omega)}) \\ &\leq M'|u-u'|(1+|u|^\beta + |u'|^\beta), \end{aligned}$$

with $M' := M + 2||u_0||_{L^\infty(\Omega)} + 2$.

Convergence of the solutions. Let (P_i) be the same problem as in (P), except that G is replaced by

$$G_i(x,u) := G \star \rho_{1/i} + |u - u_0(x)|^2.$$

The two previous paragraphs show that $(HG)'$ is satisfied for G_i with the same data as for G except that:

$$Q_i = (1+1/i)^\delta Q \quad , \quad R_i = q/i^2 + (1+1/i)^\delta + R(x),$$

$$M_i = 2 + 2||u_0||_{L^\infty(\Omega)} + 3(1+1/i)^\beta M.$$

It is clear that $||Q_i||_{L^t}, ||R_i||_{L^1}, M_i$ can be bounded independently of i.

Let u_i be a minimizer of the problem (P_i). Theorem B.1 asserts the existence of such an u_i. Moreover, the sequence (u_i) is bounded independently of i in $L^\infty(\Omega)$ and in $(W_0^{1,2}(\Omega) + \phi)$. Then, up to a subsequence, we can assume that the sequence (u_i) converges to some $u \in L^\infty(\Omega) \cap (W_0^{1,2}(\Omega) + \phi)$, weakly in $W^{1,2}(\Omega)$, strongly in $L^2(\Omega)$ and almost everywhere.

If T is a constant such that $||u_i||_{W^{1,2}(\Omega)} \le T, ||u_i||_{L^\infty(\Omega)} \le T$, then

$$\begin{aligned}|G \star \rho_{1/i}(x, u_i) - G(x, u)| &\le |G \star \rho_{1/i}(x, u_i) - G(x, u_i)| + |G(x, u_i) - G(x, u)| \\ &\le M/i(1 + 3^\beta(T+1)^\beta) + M(1 + 2T^\beta)|u_i - u| \\ &\le M(1 + 3^\beta(T+1)^\beta)(1/i + |u_i - u|).\end{aligned}$$

Whence,

$$\int_\Omega |G_i(x, u_i) - |u_i - u_0|^2 - G(x, u)| \le M(1 + 3^\beta(T+1)^\beta)(|\Omega|/i + \int_\Omega |u_i - u|),$$

which implies that $\int_\Omega G_i(x, u_i)$ converges to $\int_\Omega G(x, u) + ||u - u_0||^2_{L^2(\Omega)}$.

The map $v \mapsto \displaystyle\int_\Omega F(\nabla v)$ is lower semicontinuous (thanks to Fatou's Lemma) and convex on $W^{1,2}(\Omega)$, so that it is weakly lower semicontinuous. Hence,

$$\int_\Omega F(\nabla u) \le \text{liminf}_{\epsilon \to 0} \int_\Omega F(\nabla u_\epsilon).$$

To sum up,

$$\text{liminf}_{i \to \infty} I_i(u_i) \ge I(u) + ||u - u_0||^2_{L^2(\Omega)}.$$

Here, $I_i(v) := \int_\Omega F(\nabla v) + G_i(x, v)$. Since $I_i(u_0) \ge I_i(u_i)$ for any $i \ge 1$, we have $I(u_0) \ge I(u) + ||u - u_0||^2_{L^2(\Omega)}$. But u_0 minimizes I. Hence, $u = u_0$. This shows that u_i converges to u weakly in $W^{1,2}(\Omega)$, strongly in $L^2(\Omega)$ and almost everywhere.

Conclusion: Apply Theorem 2.1 to each problem (P_i). This is possible because each $G_i(x, \cdot)$ is smooth. Then, on each compact subset of Ω, u_i is Lipschitz of rank some V_i which can be bounded independently of i. Indeed, the proof of Theorem 2.1 shows that we can take

$$V_i := ||u_i||_{L^\infty(\Omega)} + ||\phi||_{L^\infty(\Gamma)} + C_0 g_i + C_1,$$

where C_0 and C_1 only depend on n, Ω, μ, ϕ and where g_i is the supremum of $||G_{iu}||$ on

$$\Omega \times [-||u_i||_{L^\infty(\Omega)} - 1, ||u_i||_{L^\infty(\Omega)} + ||\phi||_{L^\infty(\Gamma)} + K \text{diam}\,\Omega].$$

This supremum can be bounded independently of i by

$$M(1 + 2(T + 1 + ||\phi||_{L^\infty(\Gamma)} + K \text{diam}\,\Omega)^\beta).$$

Hence, u is also locally Lipschitz on Ω. This completes the proof of Theorem 2.3.

$\square$

B.2 Proof of Theorem B.1

Theorem B.1 is implied by Theorem 8.1 in [84].

Under the hypotheses of Theorem B.1, we have:

Theorem B.2 *There exists a minimizer in* $W_0^{1,2}(\Omega) + \phi$.

Proof: This is a routine direct method. Let (u_n) be a minimizing sequence. It is enough to show that (u_n) is bounded in $W^{1,2}(\Omega)$. Indeed, if we show this, then up to a subsequence, it converges to some $u \in W_0^{1,2}(\Omega) + \phi$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^{2^*}(\Omega)$, and almost everywhere. Then, the inequality

$$|G(x,u) - G(x,u_n)| \leq M|u - u_n|(1 + |u|^\beta + |u_n|^\beta)$$

shows that $G(x,u_n)$ converges to $G(x,u)$ almost everywhere, which implies (by Fatou's Lemma) that

$$\int_\Omega G(x,u) \leq \int_\Omega \liminf_{n\to\infty} G(x,u_n).$$

An argument already used shows that:

$$\int_\Omega F(\nabla u) \leq \liminf_{n\to\infty} \int_\Omega F(\nabla u_n).$$

This will complete the proof. So, let us show that (u_n) is bounded in $W^{1,2}(\Omega)$.

The inequality $I(u_n) \leq I(\bar{u})$, which is true if n is big enough, and hypotheses (B.2) and (B.1) show that

$$-q\int_\Omega |u_n|^2 - Q(x)|u_n|^\delta - R(x) - N|\Omega| + \mu_1 \int_\Omega |\nabla u_n|^2 \leq I(\bar{u})$$

which implies:

$$\mu_1 \int_\Omega |\nabla u_n|^2 \leq L + \int_\Omega q|u_n|^2 + Q(x)|u_n|^\delta$$

where $L := I(\bar{u}) + ||R||_{L^1(\Omega)} + N|\Omega|$.

Note that

$$\begin{aligned}
||u_n||^2_{L^2(\Omega)} &\leq (1+\epsilon)^2||u_n - \bar{u}||^2_{L^2(\Omega)} + (1 + \frac{1}{\epsilon})^2||\bar{u}||^2_{L^2(\Omega)} \\
&\leq (1+\epsilon)^2||\nabla(u_n - \bar{u})||^2_{L^2(\Omega)}/\Lambda + (1 + \frac{1}{\epsilon})^2||\bar{u}||^2_{L^2(\Omega)} \\
&\leq (1+\epsilon)^4||\nabla u_n||^2_{L^2(\Omega)}/\Lambda + (1+\epsilon)^2(1 + \frac{1}{\epsilon})^2||\nabla\bar{u}||^2_{L^2(\Omega)}/\Lambda \\
&\quad +(1 + \frac{1}{\epsilon})^2||\bar{u}||^2_{L^2(\Omega)} \\
&\leq (1+\epsilon)^4||\nabla u_n||^2_{L^2(\Omega)}/\Lambda + C(\epsilon),
\end{aligned}$$

where $C(\epsilon) = (1+\epsilon)^2/\epsilon^2(1 + (1+\epsilon)^2/\Lambda)||\bar{u}||^2_{W^{1,2}(\Omega)}$.

Note also that Holder's inequality and Sobolev's inequality imply:

$$\begin{aligned}
\int_\Omega Q(x)|u_n|^\delta &\leq ||Q||_{L^t}||u_n||^\delta_{L^{2^*}} \\
&\leq ||Q||_{L^t}(||u_n-\bar{u}||_{L^{2^*}}+||\bar{u}||_{L^{2^*}})^\delta \\
&\leq ||Q||_{L^t}(S_2||\nabla(u_n-\bar{u})||_{L^2}+||\bar{u}||_{L^{2^*}})^\delta \\
&\leq ||Q||_{L^t}(S_2||\nabla u_n||_{L^2}+S_2||\nabla\bar{u}||_{L^2}+||\bar{u}||_{L^{2^*}})^\delta \\
&\leq ||Q||_{L^t}(2^\delta S_2^\delta||\nabla u_n||^\delta_{L^2}+2^\delta C^\delta) \\
&\leq \epsilon||\nabla u_n||^2_{L^2}+C(\epsilon,||Q||_{L^t}),
\end{aligned}$$

with $C := S_2||\nabla\bar{u}||_{L^2}+||\bar{u}||_{L^{2^*}}$ and

$$C(\epsilon,||Q||_{L^t}) := \frac{((2S_2)^\delta||Q||_{L^t})^{2/(2-\delta)}}{2/(2-\delta)(2\epsilon/\delta)^{\delta/(2-\delta)}}+||Q||_{L^t}2^\delta C^\delta.$$

We have used the identity $|ab| \leq |a|^r/r+|b|^{r'}/r'$, with $r'=r/(r-1)$.

Then

$$\{\mu_1-q(1+\epsilon)^4/\Lambda-\epsilon\}\int_\Omega|\nabla u_n|^2 \leq L+C(\epsilon)+C(\epsilon,||Q||_{L^t}).$$

This shows that

$$\int_\Omega|\nabla u_n|^2 \leq C(||Q||_{L^t}),$$

with

$$C(||Q||_{L^t}) := 2\frac{L+C(\epsilon)+C(\epsilon,||Q||_{L^t})}{\mu_1-q/\Lambda}$$

if we pick $\epsilon>0$ so that $\mu_1-q(1+\epsilon)^4/\Lambda-\epsilon>(\mu_1-q/\Lambda)/2$. Hence, (∇u_n) is bounded in $L^2(\Omega)$, so that (thanks to Poincaré's inequality), (u_n) is bounded in $W^{1,2}(\Omega)$.

Theorem B.3 *Any minimizer of I on $W_0^{1,2}(\Omega)+\phi$ is bounded in $W^{1,2}(\Omega)$ by a constant T which is polynomial with respect to $||R||_{L^1}, M, ||Q||_{L^t}$.*

This is clear in view of the proof of the previous theorem.

Theorem B.4 *Any minimizer of I on $W_0^{1,2}(\Omega)+\phi$ is bounded on Ω by a constant T which is polynomial with respect to $||R||_{L^1}, M, ||Q||_{L^t}$.*

Proof: Let u be a minimizer of I on $W_0^{1,2}(\Omega)+\phi$.

Step 1: Let $t \geq k := \max(1,||\phi||_{L^\infty(\Gamma)})$. We define $A(=A(t)) := \{x\in\Omega : u(x)\geq t\}$ and $u_t := \min(u,t)\in W_0^{1,2}(\Omega)+\phi$. Then $I(u)\leq I(u_t)$ implies

$$\int_A F(\nabla u)-\int_A F(0) \leq \int_A G(x,t)-G(x,u).$$

The right hand side is lower than

$$\begin{aligned}
M\int_A(u-t)(1+|u|^\beta+t^\beta) &\leq M\int_A u(1+2|u|^\beta) \\
&\leq 3M\int_A|u|^{\beta+1}
\end{aligned}$$

while the left hand side is no lower than

$$\mu_1 \int_A |\nabla u|^2 - N|A|.$$

Finally,

$$\int_A |\nabla u|^2 \leq 3M/\mu_1 \int_A |u|^{\beta+1} + N/\mu_1 |A| \leq C(M) \int_A |u|^{\beta+1}, \qquad \text{(B.4)}$$

where $C(M) := (3M+N)/\mu_1$.

Step 2: We show that u belongs to any space $L^p(\Omega), p \geq 1$.

Let $\sigma > 0$. We multiply (B.4) by $\sigma(t-k)^{\sigma-1}$, we integrate on $\int_k^{+\infty}$, and then apply Fubini's Theorem to find:

$$\begin{aligned} \int_{A(k)} (u-k)^\sigma |\nabla u|^2 &\leq C(M) \int_{A(k)} (u-k)^\sigma |u|^{\beta+1} \\ &\leq C(M) \int_{A(k)} |u|^{\sigma+\beta+1}. \end{aligned}$$

The integral in the right hand side is finite if $\sigma+\beta+1 < 2^*$, since $u \in W^{1,2}(\Omega) \subset L^{2^*}(\Omega)$. In the left hand side, we remark that on $A(k)$,

$$(u-k)^\sigma |\nabla u|^2 = \frac{1}{(\sigma/2+1)^2} |\nabla (max(u-k,0)^{\sigma/2+1})|^2.$$

Sobolev's Lemma shows that

$$\begin{aligned} ||(u-k)^{1+\sigma/2}||_{L^{2^*}(A(k))} &\leq S_2 ||\nabla (max(u-k,0))^{\sigma/2+1}||_{L^2(\Omega)} \\ &= S_2(\sigma/2+1) ||(u-k)^\sigma |\nabla u|^2||^{1/2}_{L^1(A(k))}. \end{aligned}$$

Whence

$$\{\int_{A(k)} (u-k)^{2^*(1+\sigma/2)}\}^{2/2^*} \leq C(M) S_2^2 (1+\sigma/2)^2 \int_{A(k)} |u|^{\sigma+\beta+1}.$$

This implies

$$\begin{aligned} ||u||_{L^{2^*(1+\sigma/2)}(A(k))} &\leq ||u-k||_{L^{2^*(1+\sigma/2)}(A(k))} + k|A(k)|^{1/(2^*(1+\sigma/2))} \\ &\leq C(M,\sigma) ||u||^{a(\sigma)}_{L^{\sigma+\beta+1}(A(k))} + b(\sigma). \end{aligned}$$

with

$$\begin{gathered} a(\sigma) := (\sigma+\beta+1)/(2+\sigma)\ ,\ b(\sigma) := k|\Omega|^{1/(2^*(1+\sigma/2))} \\ \text{and } \ C(M,\sigma) := (C(M)^{1/2} S_2(\sigma/2+1))^{2/(2+\sigma)}. \end{gathered}$$

Let $\sigma_0 = 0$ and $\sigma_l > 0$ such that

$$2^*(1+\sigma_{l-1}/2) = \beta + 1 + \sigma_l.$$

The sequence (σ_l) is increasing and converges to $+\infty$. Then, by induction, we have: For each $l \geq 1$, there exists $C(l, ||u||_{L^{2^*}(\Omega)}, M)$ such that:

$$||u||_{L^{2^*(1+\sigma_l/2)}(A(k))} \leq C(l, ||u||_{L^{2^*}(\Omega)}, M).$$

Then, for each $p \geq 1$, there exists $C(p, ||u||_{L^{2^*}(\Omega)}, M)$ such that:

$$||u||_{L^p(A(k))} \leq C(p, ||u||_{L^{2^*}(\Omega)}, M).$$

This proves our claim since on $\Omega \setminus A(k), u \leq k$.

Step 3: We now prove that u is bounded thanks to (B.4). Here, A is the abbreviation of $A(t)$. For any $t \geq k$,

$$\begin{aligned}
||u-t||_{L^1(A)} &\leq ||u-t||_{L^{2^*}(A)}|A|^{1-1/2^*} \quad \text{(by Hölder's inequality)} \\
&\leq S_2||\nabla u||_{L^2(A)}|A|^{1-1/2^*} \quad \text{(by Sobolev's Lemma)} \\
&\leq S_2C(M)^{1/2}||u||^{(\beta+1)/2}_{L^{p(\beta+1)}(A)}|A|^{1/2(1-1/p)}|A|^{1-1/2^*} \\
&\qquad \text{(by Holder's inequality in (B.4))} \\
&\leq S_2C(M)^{1/2}||u||^{(\beta+1)/2}_{L^{p(\beta+1)}(A)}|A|^{1-1/2^*+1/2-1/(2p)}.
\end{aligned}$$

This completes the proof if we choose p satisfying:

$$-1/2^* + 1/2 - 1/2p = 1/n - 1/2p > 0,$$

that is $p > 2n$. Indeed, in that case,

$$||u-t||_{L^1(A)} \leq S_2C(M)^{1/2}||u||^{(\beta+1)/2}_{L^{p(\beta+1)}(A)}|A|^{\alpha},$$

with $\alpha > 1$. The previous steps show that $S_2C(M)^{1/2}||u||^{(\beta+1)/2}_{L^{p(\beta+1)}(A)}$ can be bounded by a constant $C = C(p, M, ||u||_{L^{2^*}(\Omega)})$ and thanks to Theorem B.3, $||u||_{L^{2^*}(\Omega)}$ can be bounded by a polynomial on $M, ||R||_{L^1}, ||Q||_{L^t}$. Then, the function $\rho(t) := |A(t)|$ satisfies (thanks to Fubini's Theorem):

$$\int_{t_0}^{+\infty} \rho(t)\,dt = ||u-t_0||_{L^1(A(t_0))} \leq C\rho(t_0)^{\alpha}, \quad \forall t_0 \geq k.$$

Lemma 3 in Chapter 2 implies that $\rho(t) = 0$ for any $t \geq t_1$ where

$$t_1 := k + C|\Omega|^{\alpha-1}\alpha/(\alpha-1).$$

This shows that $u \leq t_1$ and similarly we could derive a lower bound. This completes the proof of Theorem B.1.

Remark B.1 *When $n = 2$, the Sobolev embeddings show that $u \in L^p(\Omega)$ for any $p \geq 1$, (since $u \in W^{1,2}(\Omega)$). That is why we can replace 2^* by any number greater than 2 throughout this note.*

Appendix C

Appendix to Chapter 2: Some further results and open problems

C.1 Local Hölder continuity of solutions

In this section, we study the same problem as in Chapter 2 and we use the same notations. The main result of Chapter 2 requires that F is uniformly convex (this is the content of (HF)). In contrast, we only assume here that

$$(HF') \quad \text{For some } \mu > 0, \alpha \in [2, 2n/(n-1)), F \text{ satisfies,}$$

$$\text{for all } \theta \in (0,1) \text{ and } p, q \in \mathbb{R}^n :$$

$$\theta F(p) + (1-\theta)F(q) \geq F(\theta + (1-\theta)q) + (\mu/2)\theta(1-\theta)|p-q|^\alpha.$$

The condition (HF') implies that for any $p, q \in \mathbb{R}^n$ and any convex subgradient $\zeta \in \partial F(q)$ of the convex subdifferential of F at q, we have

$$\theta(F(p) - F(q)) \geq F(\theta p + (1-\theta)q) - F(q) + (\mu/2)\theta(1-\theta)|p-q|^\alpha$$

$$\geq \theta\langle\zeta, p-q\rangle + (\mu/2)\theta(1-\theta)|p-q|^\alpha,$$

which implies that

$$F(p) - F(q) \geq \langle\zeta, p-q\rangle + (\mu/2)|p-q|^\alpha. \tag{C.1}$$

We remark that when $\alpha = 2$, the condition (HF') is exactly (HF) while it is different from (HF) when $\alpha > 2$. For instance, $F(p) = |p|^3$ does not satisfy (HF) (since $\nabla^2 F(0) = 0$) but satisfies (HF') with $\alpha = 3$. To see this, remark that (HF') is true (for some $\mu > 0$) if there exists $C > 0$ such that

$$\langle\nabla F(p) - \nabla F(q), p-q\rangle \geq C|p-q|^\alpha \tag{C.2}$$

(and conversely, (C.2) implies (HF')). If $F(p) = |p|^3$, (C.2) becomes

$$\langle|p|p - |q|q, p-q\rangle \geq (C/3)|p-q|^3. \tag{C.3}$$

To prove (C.3), we may assume that $n = 2$, $p = (1, 0)$ and $q = (r\cos\theta, r\sin\theta)$. Then, it is sufficient to check that

$$(1 - r^2\cos\theta)(1 - r\cos\theta) + (-r^2\sin\theta)(-r\sin\theta) \geq C(r^2 - 2r\cos\theta + 1)^{3/2}$$

for some $C > 0$ sufficiently small, or equivalently

$$(r^3 + \beta r^2 + \beta r + 1)^2 \geq C^2(r^2 + 2\beta r + 1)^3,$$

(with $\beta = -\cos\theta \in [-1, 1]$). One can easily verify that

$$(r^3 + \beta r^2 + \beta r + 1)^2 - \frac{1}{10}(r^2 + 2\beta r + 1)^3 \geq 0$$

for any $r \geq 0$ and $\beta \in [-1, 1]$. Hence, $F(p) = |p|^3$ satisfies (HF').

We replace the hypothesis $(H\Omega)$ on Ω by

$(H\Omega')$ Ω is an open bounded set which is uniformly convex.

We recall that *uniformly convex* means that there exists $\epsilon > 0$ such that for any $x \in \Gamma$, there exists a unit vector b_x such that

$$\langle b_x, y - x\rangle \geq \epsilon|y - x|^2 \quad \forall y \in \Gamma. \tag{C.4}$$

We assume that G satisfies (HG) as in Chapter 2 and that ϕ satisfies the lower bounded slope condition. We consider again the problem (P) of minimizing

$$I(w) := \int_\Omega \{F(\nabla w(x)) + G(x, w(x))\}\, dx$$

on the set of all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that $\operatorname{tr} u = \phi$.

Then we have

Theorem C.1 *Under the hypotheses* $(H\Omega'), (HF')$ *and* (HG), *and when* ϕ *satisfies the lower bounded slope condition, any solution* u *of* (P) *is Hölder continuous and we have*

$$|u(x) - u(y)| \leq C\frac{|x - y|^{1/(\alpha-1)}}{|d_\Gamma(y|x)|}\ , \forall x, y \in \Omega,$$

where C *depends on* n, Ω, α, ϕ *and* $||u||_{L^\infty(\Omega)}$.

We recall that $d_\Gamma(y|x) = |y - \pi_\Gamma(y|x)|$ where $\pi_\Gamma(y|x)$ is the unique point of Γ of the form $y + t(x - y)$ with $t \geq 0$.

The main difference between the proof of Theorem 2.1 in Chapter 2 and the proof of Theorem C.1 is the existence of a barrier. In this section, we use the barrier which appears in [84].

Proof (sketch):

The lower barrier condition The corresponding version of Chapter 2, Theorem 2.2 is

Theorem C.2 *Under hypotheses $(H\Omega'), (HF')$ and (HG), let u be a bounded solution of problem (P) as described above, where the function ϕ satisfies the lower bounded slope condition of rank K. Then there exists $\bar{K} > 0$ with the following property: for any $\bar{x} \in \Gamma$, there exists a function w which is Lipschitz of rank $\bar{K}$, which agrees with ϕ at $\bar{x}$, and which satisfies $w \leq u$ a.e. in Ω.*

Proof of Theorem C.2: The proof uses the following lemma of Stampacchia (see [84] and [47]):

Lemma C.1 *Let $x \in \Gamma$ and b_x as in (C.4). We denote by $\delta : \Omega \to \mathbb{R}$ the distance to the hyperplane H which is orthogonal to b_x and contains x. Then for any $s \in (1, (n+1)/2)$, the quantity*

$$\int_\Omega \frac{dx}{\delta(x)^s}$$

is finite and bounded from above by a constant (which only depends on s and n).

We proceed to prove Theorem C.2. Let $\bar{x} \in \Gamma$, and $b_{\bar{x}}, H, \delta$ as in Lemma C.1. Since ϕ satisfies the lower bounded slope condition of rank K, there exists $\zeta \in \bar{B}(0, K)$ such that

$$\phi(y) \geq \phi(\bar{x}) + \langle \zeta, y - \bar{x} \rangle \ , \ \forall y \in \Gamma.$$

Then, for any $d > 0$, we define

$$w_d(y) := \phi(\bar{x}) + \langle \zeta, y - \bar{x} \rangle - d\delta(y).$$

We proceed to prove that w_d has the required properties (for d sufficiently large). Clearly w_d agrees with ϕ at $\bar{x}$. Moreover, the map w_d is affine when restricted to the half space bounded by H and containing Ω and

$$\nabla w_d(y) = \zeta - db/|b| =: k.$$

We need only show that the following set has measure 0 :

$$\Lambda_d := \{x \in \Omega : w_d(x) > u(x)\}.$$

Since

$$w_d(y) \leq \phi(\bar{x}) + \langle \zeta, y - \bar{x} \rangle \leq \phi(y) \ , \ \forall y \in \Gamma,$$

the map $v := \max(u, w_d)$ satisfies $\operatorname{tr} v = \phi$. Hence $I(u) \leq I(v)$, which yields

$$\int_{\Lambda_d} \{F(\nabla u(x)) - F(k)\}\, dx \leq \int_{\Lambda_d} \{G(x, w_d(x)) - G(x, u(x))\}\, dx.$$

Since $u - v \in W_0^{1,1}(\Omega)$, we have (by Stoke's formula)

$$\int_{\Lambda_d} \{\nabla u(x) - \nabla w_d(x)\}\, dx$$

so that (using (C.1))

$$(\mu/2) \int_{\Lambda_d} |\nabla(u - w_d)(x)|^\alpha\, dx \leq \int_{\Lambda_d} \{G(x, w_d(x)) - G(x, u(x))\}\, dx.$$

The map w_d is convex and bounded from above by $||\phi||_{L^\infty(\Gamma)}$ on Γ. Hence, w_d is bounded from above on Ω. Since $u \leq w_d$ on Λ_d, we get (using (HG))

$$|G(x, w_d(x)) - G(x, u(x))| \leq L|w_d(x) - u(x)|,$$

where L is a Lipschitz rank for G on

$$[-||u||_{L^\infty(\Omega)}, ||\phi||_{L^\infty(\Gamma)}].$$

Finally,

$$(\mu/2)\int_{\Lambda_d} |\nabla u(x) - k|^\alpha \, dx \leq L \int_{\Lambda_d} |w_d(x) - u(x)| \, dx. \tag{C.5}$$

Now, since $\alpha < \frac{2n}{n-1}$, we have

$$1 - \frac{1}{n(\alpha-1)} < \frac{n^2+1}{n^2+n},$$

so that there exists $1 \leq l < n$ which satisfies

$$1 - \frac{1}{n(\alpha-1)} < \frac{1}{l} < \frac{n^2+1}{n^2+n}.$$

Moreover, we can find such an l in $[1, \alpha]$ since $\frac{n^2+n}{n^2+1} < 2 \leq \alpha$.

We now apply Sobolev's and Hölder's inequalities to $u - v \in W_0^{1,1}(\Omega)$ on the left hand side. We get:

$$\{\int_{\Lambda_d} |u - w_d|^{l^*}\}^{1/l^*} \leq S_l |\Lambda_d|^{1/l - 1/\alpha} \{(2L/\mu)\int_{\Lambda_d} |w_d - u|\}^{1/\alpha}.$$

Here, the constant S_l is defined by

$$1/S_l := \inf_{w \in W_0^{1,\alpha}(\Omega)} \frac{||Dw||_{L^\alpha(\Omega)}}{||w||_{L^{\alpha^*}(\Omega)}}$$

(where $1/l^* = 1/l - 1/n$).

Hölder's inequality on the right hand side yields

$$\{\int_{\Lambda_d} |u - w_d|^{l^*}\}^{1/l^*} \leq S_l (2L/\mu)^{1/\alpha} \{\int_{\Lambda_d} |w_d - u|^{l^*}\}^{1/(\alpha l^*)} |\Lambda_d|^{1/l - 1/(\alpha l^*)}$$

and finally

$$||u - w_d||_{L^{l^*}(\Lambda_d)} \leq C|\Lambda_d|^{\alpha(1/l - 1/(\alpha l^*))/(\alpha-1)} \tag{C.6}$$

where $C := S_l^{\alpha/(\alpha-1)}(2L/\mu)^{1/(\alpha-1)}$.

We now introduce $\rho(d) := |\Lambda_d|$. We have for any $d > 0$:

$$\int_d^{+\infty} \rho(e) \, de = \int_{\Lambda_d} \frac{w_d(x) - u(x)}{\delta(x)} \, dx$$

$$\leq (\int_{\Lambda_d} |u(x) - w_d(x)|^{l^*})^{1/l^*} (\int_{\Lambda_d} \frac{dx}{\delta(x)^s})^{1/s}$$

where s is defined by $1/l^* + 1/s = 1$. Since $1/l < (n^2+1)/(n^2+n)$, we have $s < (n+1)/2$ so that (by Lemma C.1),

$$(\int_{\Lambda_d} \frac{dx}{\delta(x)^s})^{1/s} < \infty.$$

Using (C.6), we then get

$$\int_d^{+\infty} \rho(x)\,dx \le C'\rho(d)^\gamma$$

with $\gamma := \dfrac{\alpha}{\alpha-1}(\dfrac{1}{l} - \dfrac{1}{\alpha l^*})$. Since $1/l > 1 - 1/(n(\alpha-1))$, we have $\gamma > 1$.

Chapter 2, Lemma 3 shows that $\rho(d) = 0$ for any $d \ge d_0$ where

$$d_0 := C'|\Omega|^{\gamma-1}\frac{\gamma}{\gamma-1}.$$

For these values of d, $w_d \le u$ on Ω, which completes the proof of Theorem C.2.

The end of the proof of Theorem C.1 This part is very similar to Chapter 2, section 2.2.2 so that we only indicate the minor changes in the proof. We define Ω_λ and u_λ as in Chapter 2. Chapter 2, Lemma 1 and its proof as well as the inequalities leading to Chapter 2, (2.5) remain true without any change. From Chapter 2, (2.5), we use (HF') to get Chapter 2, (2.6) except that the number 2 is replaced by α. Thanks to Chapter 2, Lemma 2, we have the following version of Chapter 2, (2.9):

$$\mu||\nabla(u_\lambda - u)||^{\alpha-1}_{L^\alpha(A)} \le [||f||_\infty + g_0 C_p(\frac{1}{\lambda^n} - \frac{1}{\lambda})]|A|^{1-1/\alpha}. \qquad \text{(C.7)}$$

with the same notations as in Chapter 2. From (C.7), we get as in Chapter 2 (using Hölder's and Sobolev's inequalities):

$$||u_\lambda - u||_{L^1(A)} \le C[||f||_\infty + (\frac{1}{\lambda^n} - \frac{1}{\lambda})]^{\frac{1}{\alpha-1}}|A|^\gamma$$

with $\gamma := 1 + 1/n$. From this estimate, we derive (for the same reasons as in Chapter 2):

$$||u_\lambda - u||_{L^1(A)} \le C_2(1-\lambda)^{1/(\alpha-1)}|A|^\gamma.$$

Chapter 2, (2.10) becomes

$$\int_q^{+\infty} \rho(q')\,dq' \le C_2(1-\lambda)^{\frac{2-\alpha}{\alpha-1}}|\rho(q)|^\gamma$$

for any

$$q > \bar{q} := \bar{K}\operatorname{diam}\Omega + ||\phi||_{L^\infty(\Gamma)}.$$

Chapter 2, Lemma 3 then implies that $|A(q)| = 0$ if

$$q \ge q_0 := C_2(1-\lambda)^{\frac{2-\alpha}{\alpha-1}}\frac{\gamma}{\gamma-1}|\Omega|^{\gamma-1} + \bar{q}.$$

There exists $\lambda_0 \in (1/2, 1)$ such that for any $\lambda \in (\lambda_0, 1)$, we have

$$C_2(1-\lambda)^{\frac{2-\alpha}{\alpha-1}}\frac{\gamma}{\gamma-1}|\Omega|^{\gamma-1} \ge \bar{q}.$$

Then $A(q) = 0$ for any $q \geq q_1 := C_3(1-\lambda)^{\frac{2-\alpha}{\alpha-1}}$ with $C_3 := 2C_2 \dfrac{\gamma}{\gamma-1}|\Omega|^{\gamma-1}$ and any $\lambda \geq \lambda_0$. Hence, for any choice of $z \in \Gamma$, we have, almost everywhere on Ω_λ, the inequality

$$\begin{aligned} u_\lambda(x) &:= \lambda u((x-z)/\lambda + z) - q_1(1-\lambda) \\ &= \lambda u((x-z)/\lambda + z) - C(1-\lambda)^{\frac{2-\alpha}{\alpha-1}+1} \leq u(x) \end{aligned}$$

(where C does not depend on $\lambda \in (\lambda_0, 1)$). In the final step of the proof, we consider (as in Chapter 2) two Lebesgue points x and y of u such that

$$x \in B(y, (1-\lambda_0)d_\Gamma(y)).$$

This last condition ensures that $\lambda \in (\lambda_0, 1)$ where λ is defined by

$$y = \frac{x-z}{\lambda} + z \quad , \quad z := \pi_\Gamma(y|x).$$

Then the analogue of Chapter 2, (2.11) is

$$u(y) \leq u(x) + Q\frac{|x-y|^{\frac{1}{\alpha-1}}}{|y-\pi_\Gamma(y|x)|^{\frac{1}{\alpha-1}}}.$$

This completes the proof of Theorem C.1.

□

C.2 An Hilbert-Haar theory on nonconvex sets

In the classical Hilbert-Haar theorem, the bounded slope condition implies that the open bounded subset Ω of $\mathbb{R}^n$ is convex. In contrast, the lower bounded slope condition does not imply that Ω is convex and yet, it is assumed in [28] (and in [13]) that Ω is convex. In this section, we investigate the case of an open bounded convex set with a hole. More precisely, consider an open bounded convex set Ω_1 and a map $\phi_1 : \Gamma_1 := \partial\Omega_1 \to \mathbb{R}$ which satisfies the lower bounded slope condition. Let Ω_2 be an open bounded set of class $C^{1,1}$ which satisfies:

$$\bar{\Omega}_2 \subset \Omega_1.$$

Let $\phi_2 : \Gamma_2 := \partial\Omega_2 \to \mathbb{R}$ be a map of class C^2. Then we can define $\Omega := \Omega_1 \setminus \Omega_2$, $\Gamma := \partial\Omega$ and the map $\phi : \Gamma := \partial\Omega \to \mathbb{R}$ such that

$$\phi|_{\Gamma_1} = \phi_1 \qquad \phi|_{\Gamma_2} = \phi_2.$$

Then, ϕ is the trace of a map in $W^{1,1}(\Omega)$ still denoted by ϕ.

We introduce the Lagrangian F which is assumed to be a C^2 convex map. We denote by E the Bernstein function: $E(p) := \langle \nabla^2 F(p)p, p\rangle$ and $\Lambda(p)$ ($\lambda(p)$) the biggest (lowest) eigenvalue of $\nabla^2 F(p)$. Let us assume that $\lambda(p) > 0$ for any $p \in \mathbb{R}^n$ and that

$$\limsup_{|p|\to\infty} \frac{|p|\Lambda(p)}{E(p)} < +\infty.$$

We consider

$$I : u \in W^{1,1}(\Omega) \mapsto \int_\Omega F(\nabla u(x))\, dx$$

and we study the problem (P) of minimizing I on $W_0^{1,1}(\Omega) + \phi$.

Theorem C.3 *Let u be a solution of (P). Then the map u is locally Lipschitz on Ω.*

Proof: The proof is almost the same as the proof of [28] except that Ω_λ is now replaced by $\Omega \cap \Omega_\lambda$. Once again, the key point is the existence of a barrier:

Lemma C.2 *i) There exists a Lipschitz map $w : \bar{\Omega} \to \mathbb{R}$ such that*

$$w|_\Gamma = \phi \ , \ w \leq u \ \ \textit{a.e. on} \ \ \Omega. \tag{C.8}$$

ii) There exists a Lipschitz map $v : \bar{\Omega} \to \mathbb{R}$ such that

$$v|_{\Gamma_2} = \phi_2 \ , \ v \geq u \ \ \textit{a.e. on} \ \ \Omega. \tag{C.9}$$

We admit Lemma C.2 for the moment and we proceed to prove Theorem C.3.

Step 1: For any $\lambda \in (0,1)$, $z \in \bar{\Omega}$, we define

$$\Omega_\lambda := \lambda(\Omega - z) + z \ , \ \Gamma_\lambda := \partial\Omega_\lambda$$

$$u_\lambda(x) := \lambda u(\frac{x-z}{\lambda} + z) \ , \ x \in \Omega_\lambda.$$

We proceed to show that there exists $Q_1 \geq 0$ such that

$$u_\lambda \leq u + Q_1(1-\lambda) \ \ \text{on} \ \ \partial(\Omega \cap \Omega_\lambda). \tag{C.10}$$

As in [13], we may assume (only for this step) that $u \in C^0(\bar{\Omega})$. In contrast with [28], we can not assert that $\partial(\Omega \cap \Omega_\lambda) = \Gamma_\lambda$. However, we have:

$$\partial(\Omega \cap \Omega_\lambda) \subset \Gamma_\lambda \cup \Gamma_2. \tag{C.11}$$

Let us justify this inclusion. Let $x \in \partial(\Omega \cap \Omega_\lambda)$ and assume that $x \notin \Gamma_\lambda$. Then $x \in \Omega_\lambda \setminus \Omega$, which implies that there exists $y \in \Omega$ such that

$$x = \lambda(y - z) + z.$$

Since $x \in \bar{\Omega} \setminus \Omega$, we have $x \in \Gamma_1 \cup \Gamma_2$. Suppose that x fails to be in Γ_2. Then $y \in \Omega_1$, $z \in \bar{\Omega}_1$, $x \in \Gamma_1$ and $x \in (z,y)$, which is impossible since Ω_1 is convex. Then $x \in \Gamma_2$ and the inclusion (C.11) is proved.

Let $x \in \partial(\Omega \cap \Omega_\lambda)$. Then we have to consider two different cases: $x \in \Gamma_\lambda$ and $x \in \Gamma_2 \setminus \Gamma_\lambda$.

If $x \in \Gamma_\lambda$, then $y \in \Gamma$, where $y := \dfrac{x-z}{\lambda} + z$ so that $w(y) = u(y)$. Moreover, $w(x) \leq u(x)$. Finally,

$$u(x) \geq w(x) \geq w(y) - Q|x-y| \geq u(y) - Q(1-\lambda)\operatorname{diam}\Omega.$$

Here, Q denotes the Lipschitz rank of w. This easily implies (C.10) with $Q_1 = Q\operatorname{diam}\Omega + ||\phi||_{L^\infty(\Gamma)}$ (here, we use the fact that u is bounded by $||\phi||_{L^\infty(\Gamma)}$, which can be proved as in [28]).

If $x \in \Gamma_2$, we use the fact that $v(y) \geq u(y)$ since $y \in \bar{\Omega}$. Hence,

$$u(x) = \phi_2(x) = v(x) \geq v(y) - Q'|x-y| \geq u(y) - Q'(1-\lambda)\operatorname{diam}\Omega.$$

Here, Q' denotes the Lipschitz rank of v. This easily implies (C.10) with $Q_1 = Q'\operatorname{diam}\Omega + ||\phi||_{L^\infty(\Gamma)}$.

In any case, (C.10) is true with $Q_1 := \max(Q, Q')\operatorname{diam}\Omega + ||\phi||_{L^\infty(\Gamma)}$.

Step 2: The map u_λ is a minimizer on Ω_λ. Furthermore, the maps $u+Q_1(1-\lambda)$ and u_λ are two minimizers on $\Omega\cap\Omega_\lambda$. Using Step 1 and the comparison Principle (see [62], Theorem 4.1), we get

$$u_\lambda(y) \leq u(y) + Q_1(1-\lambda) \text{ a.e. } y \in \Omega \cap \Omega_\lambda.$$

Step 3: Fix two distinct Lebesgue points x and y in Ω. The half line $\{y + t(x-y) : t \geq 0\}$ may intersect Γ in several points z but one of them must satisfy $x \in (y,z)$ (this is the case if we choose $z \in \Gamma_1$). Hence there exists $\lambda \in (0,1)$ such that

$$x = \lambda(y-z) + z,$$

so that $x \in \Omega \cap \Omega_\lambda$. There exists $r > 0$ such that $B(x,r) \subset \Omega \cap \Omega_\lambda$ and for almost every point $x' \in B(x,r)$, we have $u_\lambda(x') \leq u(x') + Q_1(1-\lambda)$. Then we can complete the proof as in Chapter 2 to get

$$u(y) - u(x) \leq (Q_1 + ||\phi||_{L^\infty(\Gamma)})\frac{|x-y|}{|y-z|}.$$

This shows that u is locally Lipschitz.

□

We now prove Lemma C.2. We begin with an easy result:

Proposition C.1 *There exists a Lipschitz map $w_0 : \bar{\Omega} \to \mathbb{R}$ which satisfies $w_0|_{\Gamma_1} = \phi_1$ and $w_0 \leq u$ a.e. on Ω.*

Proof: To show the existence of such a map w_0, it suffices to find some $Q \geq 0$ such that for any $x \in \Gamma_1$, there exists a Lipschitz map l_x of rank Q satisfying

$$l_x(x) = \phi(x) \quad , \quad l_x \leq u \text{ a.e. on } \Omega. \tag{C.12}$$

Indeed, if such an l_x exists for any $x \in \Gamma$, then we can define:

$$w_0 := \sup_{x\in\Gamma_1} l_x.$$

Since ϕ_1 satisfies the lower bounded slope condition, there exists $K \geq 0$ such that for any $x \in \Gamma_1$, there exists $\zeta_x \in \bar{B}(0,K)$ such that

$$\phi_1(y) \geq \phi_1(x) + \langle \zeta_x, y-x\rangle \quad \forall y \in \Gamma_1.$$

For any $x \in \Gamma_1$, we choose a unit vector n_x in the convex normal cone to Ω_1 at x. Then, for any $y \in \Gamma_2$,

$$\langle n_x, y-x\rangle < 0.$$

In fact, we can find some $\delta > 0$ such that for any $x \in \Gamma_1, y \in \Gamma_2$,

$$\langle n_x, y-x\rangle \leq -\delta.$$

Hence, we have

$$\phi(y) \geq \phi_1(x) + \langle \zeta_x + Tn_x, y-x\rangle \quad \forall y \in \Gamma$$

for some $T > 0$ if

$$\min_{\Gamma_2} \phi_2 \geq \max_{\Gamma_1} \phi_1 + K \operatorname{diam} \Omega - \delta T$$

which is equivalent to

$$T \leq \frac{\min_{\Gamma_2}\phi_2 - \max_{\Gamma_1}\phi_1 - Q_1 \mathrm{diam}\,\Omega}{\delta} =: \bar{T}.$$

We denote

$$K' := K + \bar{T}$$

and for any $x \in \Gamma_1$,

$$l_x(y) := \phi_1(x) + \langle \zeta_x + \bar{T} n_x, y - x \rangle \quad \forall y \in \mathbb{R}^n.$$

The map l_x satisfies (C.12) (here we use the Comparison Principle, see [62], Theorem 4.1 and the fact that an affine function is a minimizer). Proposition C.1 is proved.

□

Using the equivalence between *subminima* of the variational problem and *subsolutions* of the corresponding Euler equation (see [62]), we have (see the proof of [37],Theorem 1.5).

Proposition C.2 *i) There exists a Lipschitz map* $w_1 : \bar{\Omega} \to \mathbb{R}$ *such that* $w_1|_{\Gamma_2} = \phi_2$ *and* $w_1 \leq u$ *a.e. on* Ω.
ii) There exists a Lipschitz map $v : \bar{\Omega} \to \mathbb{R}$ *such that* $v|_{\Gamma_2} = \phi_2$ *and* $v \geq u$ *a.e. on* Ω.

Then the map $w := \max(w_0, w_1)$ satisfies (C.8). The map v satisfies (C.9). This completes the proof of Lemma C.2.

□

Finally, we remark that there is a corresponding version of the classical Hilbert-Haar theorem on an open convex set with a "smooth hole". The key point is the existence of a barrier (as in Lemma C.2 above). The rest of the proof is based on [62], Theorem 5.1.

C.3 A theorem on continuity

The main idea of the classical Hilbert-Haar Theorem (and of the theorem of Clarke [28] as well) is to compare a minimizer u with another minimizer which can be written as $u \circ \phi_\lambda$ where ϕ_λ is an affine map from Ω onto Ω_λ. The comparison takes place on $\Omega \cap \Omega_\lambda$. The affine map is a *translation* in the classical Hilbert-Haar Theorem and it is a *contraction* in the theorem of Clarke. In the following example, we use a *rotation* to prove the continuity of a minimizer.

Let $\Omega := B(0,R) \setminus \bar{B}(0,r)$ and $F : [0,\infty) \to \mathbb{R}$ be a nondecreasing convex function. Let $\phi : \Gamma := \partial\Omega \to \mathbb{R}$ be a continuous map. Then we consider the problem (P) of minimizing

$$I(u) := \int_\Omega F(|\nabla u(x)|)\, dx$$

on the set of those $u \in W^{1,1}(\Omega)$ such that $\mathrm{tr}\, u = \phi$.

Theorem C.4 *Let* u *be a solution of (P). Then* u *is continuous on* $\bar{\Omega}$.

Proof: Let H be a 2 plane in $\mathbb{R}^n$ containing 0. Then, for any $\theta \in \mathbb{R}$, we denote by

$$\mathcal{R}_\theta : \mathbb{R} \to \mathbb{R}$$

the map which is the rotation through angle θ on H and the identity on $H^\perp$.

We also define $u_\theta := u \circ \mathcal{R}_\theta$. Then $u_\theta \in W^{1,1}(\Omega)$. An obvious change of variables yields $I(u_\theta) = I(u)$, which implies that u_θ is a minimizer. Using the Maximum Principle (see [62], Corollary 4.1), we get the key estimate:

$$||u - u_\theta||_{L^\infty(\Omega)} \leq ||\phi - \phi_\theta||_{L^\infty(\Gamma)}, \tag{C.13}$$

where $\phi_\theta := \phi \circ \mathcal{R}_\theta$.

Let us denote by $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ the modulus of continuity of ϕ. Then we have

$$||u - u_\theta||_{L^\infty(\Omega)} \leq \omega(R|1 - e^{i\theta}|). \tag{C.14}$$

We fix a point $x_0 \in \bar{\Omega}$ and we proceed to show that u is continuous at x_0. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(2R\delta) < \epsilon$. The map u is absolutely continuous on almost every "radius": $l_x := [r\frac{x}{|x|}, R\frac{x}{|x|}], x \in \mathbb{R}^n$. Hence, there exists $x_2 \in \partial B(0, R)$ such that u is continuous on l_{x_2} and $|\frac{x_2}{|x_2|} - \frac{x_0}{|x_0|}| < \delta$. We denote by θ the angle of the rotation which maps x_0 onto $x_2|x_0|/|x_2|$ in the 2 plane containing $0, x_0, x_2$.

By continuity of u on l_{x_2}, there exists $\rho > 0$ such that for $y, y' \in l_{x_2}$, we have

$$|y - y'| < \rho \Longrightarrow |u(y) - u(y')| < \epsilon. \tag{C.15}$$

For any $x \in \bar{\Omega} \cap B(x_0, \delta|x_0|/2)$, we have

$$\begin{aligned} |u(x) - u(x_0)| \leq |u(x) - u(|x|\frac{x_2}{|x_2|})| &+ |u(|x|\frac{x_2}{|x_2|}) - u(|x_0|\frac{x_2}{|x_2|})| \\ &+ |u(|x_0|\frac{x_2}{|x_2|}) - u(x_0)|. \end{aligned} \tag{C.16}$$

In the 2 plane containing $0, x, x_2$, we denote by θ' the angle of the rotation which maps x onto $|x|\frac{x_2}{|x_2|}$. Then (C.16) and (C.14) imply

$$|u(x) - u(x_0)| \leq \omega(R|1 - e^{i\theta'}|) + |u(|x|\frac{x_2}{|x_2|}) - u(|x_0|\frac{x_2}{|x_2|})| + \omega(R|1 - e^{i\theta}|).$$

Remark that $||x|\frac{x_2}{|x_2|} - |x_0|\frac{x_2}{|x_2|}| \leq |x_0 - x| < \rho$ so that (by (C.15))

$$|u(x) - u(x_0)| \leq \omega(R|1 - e^{i\theta'}|) + \epsilon + \omega(R|1 - e^{i\theta}|).$$

Moreover, $|1 - e^{i\theta}| < \delta$ and

$$\begin{aligned} |1 - e^{i\theta'}| = |\frac{x}{|x|} - \frac{x_2}{|x_2|}| \leq |\frac{x}{|x|} - \frac{x_0}{|x_0|}| + |\frac{x_0}{|x_0|} - \frac{x_2}{|x_2|}| \leq |\frac{x}{|x|} - \frac{x_0}{|x_0|}| + \delta \\ \leq 2\frac{|x - x_0|}{|x_0|} + \delta \leq 2\delta \end{aligned}$$

This shows that $\omega(R|1 - e^{i\theta}|) < \epsilon$, $\omega(R|1 - e^{i\theta'}|) < \epsilon$. Finally, for any $x \in \bar{\Omega}$ $\cap B(x_0, \delta|x_0|/2)$, we have

$$|u(x) - u(x_0)| < 3\epsilon.$$

Hence u is continuous at x_0. This completes the proof of Theorem C.4.

□

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